

# Top 10 Symmetries Every Quantum Mechanic Should Know

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DAY 2

## Two classics, one modern gem

- Herman Weyl, *Symmetry* (1952, Princeton 2016)
- Eugene Wigner, “The Unreasonable Effectiveness of Mathematics in the Natural Sciences” (1960)
- Conway, Burgiel, Goodman-Strauss, *The Symmetry of Things* (2008)

# References

## Basic references

- Vincent Bouchard: Lecture notes
  - <https://sites.ualberta.ca/~vbouchar/MAPH464/notes.html>
- Anthony Zee: Group Theory in a Nutshell for Physicists (2016)
- H.F. Jones: Groups, Representations, and Physics (1998)
- Morton Hamermesh: Group Theory and Its Applications to Physical Problems (1962, Dover 1989)
- Michael Tinkham, Group Theory and Quantum Mechanics, (McGraw-Hill, New York, 1964)
- Eugene P. Wigner, Group Theory and Its Application to the Quantum Mechanics of Atomic Spectra, J. J. Griffin, Eng. trans., (1959)

## On special topics

- Robert Gilmore: Lie Groups, Lie Algebras, and Some of Their Applications (1974, Dover 2006)
- Brian G. Wybourne, Classical Groups for Physicists (1974)
- M. E. Rose, Elementary Theory of Angular Momentum (1957)
- A. R. Edmonds, Angular Momentum in Quantum Mechanics (1957)

# Some types of symmetries

## **Geometric:**

Symmetries that map a space onto itself. Could be physical space, configuration space, phase space, parameter space,...

## **Kinematic:**

Symmetries that commute with Hamiltonian

## **Dynamic:**

Symmetries that act on algebras of operators that includes Hamiltonian and/or preserve the action

# Top 10 Symmetries Every Quantum Mechanic Should Know

## Fundamental symmetries

- Involution
- Products of Involutions

## Geometrical Symmetries

- Point symmetry
- Rotational symmetry
- Translational symmetry
- Euclidean symmetry
- Lattice symmetry

## More general kinematic and dynamics symmetries

- Permutation symmetry
- Unitary symmetry
- Galilean symmetry

**Non-relativistic**

## Next 8 best symmetries

- Symplectic symmetry
- Conformal symmetry
- Exchange symmetries
- Lorentz symmetry
- Poincare symmetry
- Gauge symmetry
- Schrodinger symmetries
- Oscillator symmetries

# 5. Translational symmetry

Homogeneity in space and/or time

*R*

Key math concepts: non-compact group, unbounded operators

# Continuous, abelian group

- Real line with addition

$$T_x \sim (\mathbb{R}, +)$$

- Unitary abelian representation on a Hilbert space

$$U(b)U(a) = U(a)U(b) = U(a + b)$$

$$U(-a) = U^\dagger(a) = U^{-1}(a)$$

- Derivative is sometimes useful

$$P = i\hbar \frac{d}{dx} U(x) \Big|_{x=0}$$

$$U(a) = e^{-iaP/\hbar}$$

$$U(x)HU^\dagger(x) = H \Leftrightarrow [P, H] = 0$$

# Continuous, abelian group

- Real line with addition

$$T_t \sim (\mathbb{R}, +)$$

- Unitary abelian representation on a Hilbert space

$$U(t)U(t') = U(t')U(t) = U(t+t')$$

$$U(-t) = U^\dagger(t) = U^{-1}(t)$$

- Derivative is sometimes useful

$$H = i\hbar \left. \frac{d}{dt} U(t) \right|_{t=0}$$

# Scale transformation

- Consider the differential operator

$$x \frac{\partial}{\partial x} \quad \longrightarrow \quad \left( x \frac{\partial}{\partial x} \right) x^n = nx^n$$

- Dilation operator

$$e^{\alpha x \frac{\partial}{\partial x}} f(x) = f(e^\alpha x)$$

# Scale dilations in free space QM

Calogero interaction  
2-D delta function  
1-D delta prime

- Consider the free space Hamiltonian

$$H_0 = -\frac{\hbar^2}{2m} \sum_i \Delta_i + \sum_{i < j} V(|r_i - r_j|) \quad \text{with} \quad V(\lambda|r|) = \lambda^{-2}V(|r|)$$

- Scale transformation leaves Schrodinger equation invariant

$$r \rightarrow \lambda r, t \rightarrow \lambda^2 t$$

- Dilation operator

$$Q = \frac{1}{2} \sum_i (r_i p_i + p_i r_i) \quad [Q, H_0] = 2H_0$$

Can also add harmonic trapping potential

# 6. Euclidean symmetry

Homogeneity and isotropy

$E_2$ 

Key math concepts: semidirect product

## Euclidean group

$$E(N) = ISO(N)$$

- Isometries of space

$$|y' - x'| = |y - x|$$

- Matrix representation

$$x \rightarrow x' = Rx + a$$

$$R \in O(N), a \in T_N$$

$$\begin{pmatrix} x'_1 \\ x'_2 \\ 1 \end{pmatrix} = \begin{pmatrix} R_{11} & R_{12} & a_1 \\ R_{21} & R_{22} & a_2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ 1 \end{pmatrix}$$

- Semidirect product

$$E(N) = O(N) \ltimes T_N$$

$$(R', a')(R, a) = (R'R, a' + R'a)$$

$$(R', 0)(R, 0) = (R'R, 0) \quad (I, a')(I, a) = (I, a' + a)$$

Lie algebra for connected component  
and Inonu-Wigner Contraction

# Applications

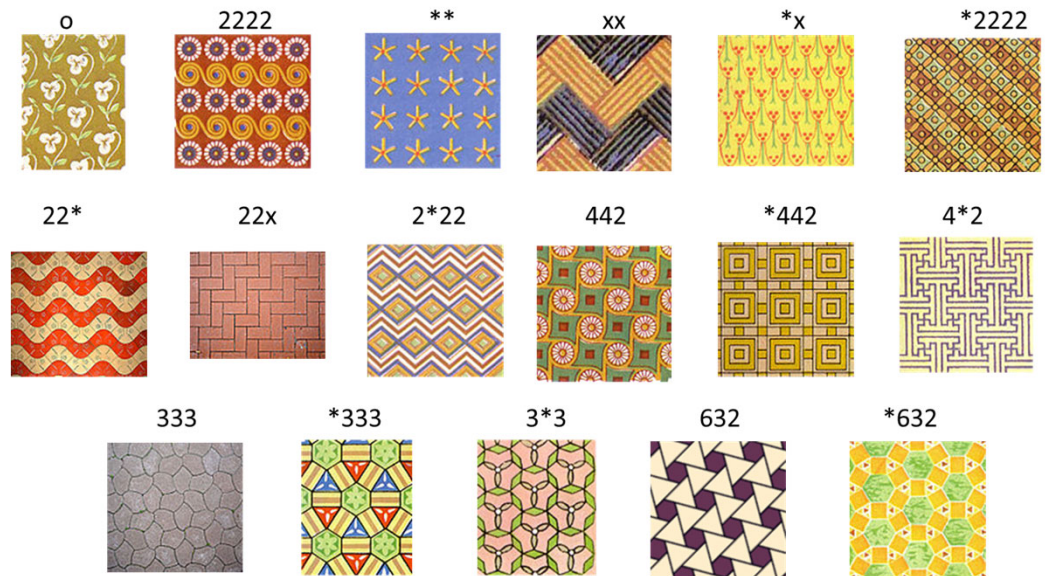
- Free space in  $N$  dimensions
  - Two and three dimensions well-studied for 2000+ years
- Subgroups include point groups and space groups
  - Classified (but not counted)
- Configuration space
- Relative configuration space
  - Rotations in space shape!
- Relativistic quantum mechanics
  - Little group of photon

# 7. Lattice symmetry

Periodicity combined with point symmetries

# Euclidean geometry and space groups

- Space groups
  - 1D: 2
  - 2D: 17 (wallpaper groups)
  - 3D: 230 (crystal groups)
  - 4D: 4894
  - 5D: 222,097
  - >5D: ???
- Line groups:
  - 1D: 2
  - 2D: 7 (frieze groups)
  - 3D:  $\infty$  (rod groups)
  - >3D:  $\infty$



From wikipedia: Wallpaper group



Couldn't decide between the two 1D lattice groups for the example:

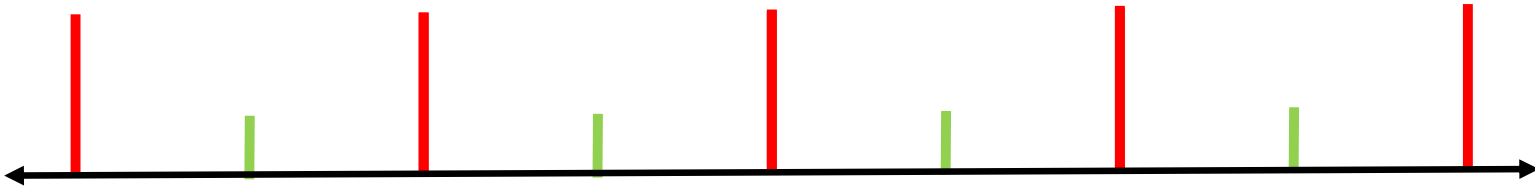
$Z$  or  $D_{\infty}$

Key math concepts: cosets, space groups, symmorphic vs non-symmorphic

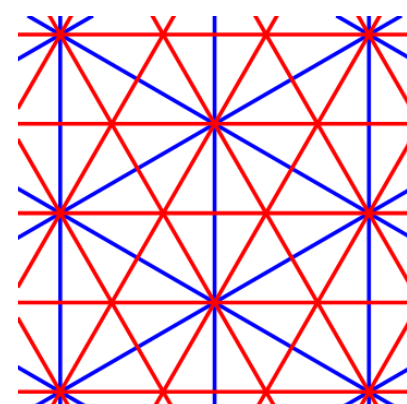
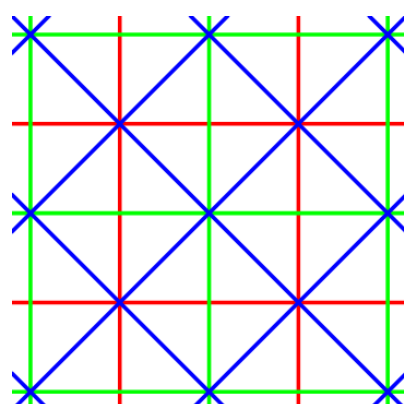
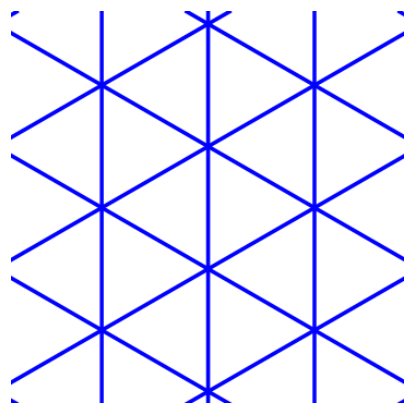
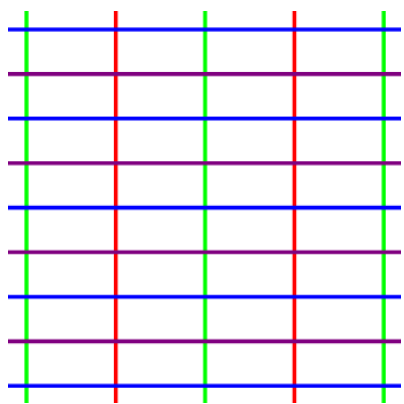
# Applications

- Crystals and Bloch's Theorem
- Periodic Hamiltonians and Floquet's Theorem

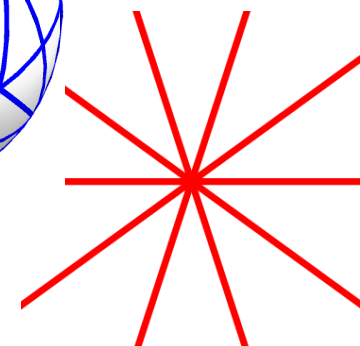
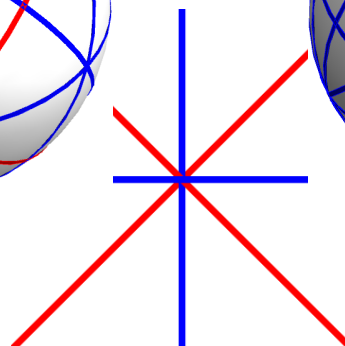
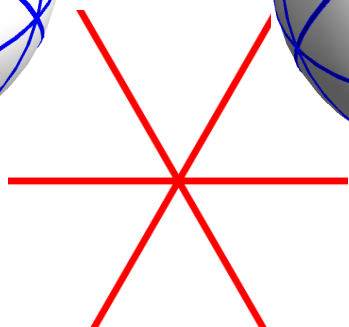
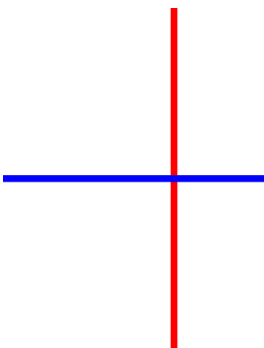
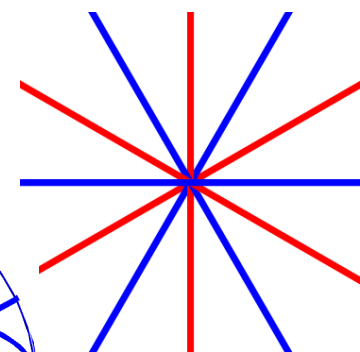
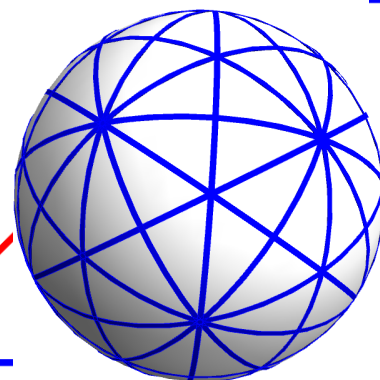
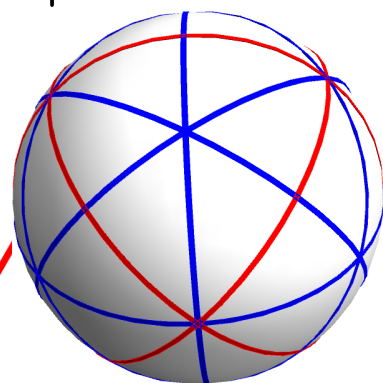
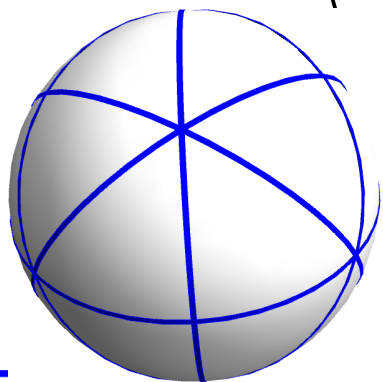
$$I_2(\infty) = \langle \sigma_1, \sigma_2 \mid \sigma_1^2 = \sigma_2^2 = e \rangle = D_\infty$$



$$H = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + \alpha \sum_{n \in \mathbb{Z}} \delta(x - nL) + \beta \sum_{i \in \mathbb{Z}} \delta\left(x - \frac{(2i+1)}{2} L\right)$$



$$\left\langle \sigma_1, \dots, \sigma_r \mid \sigma_i^2 = (\sigma_i \sigma_j)^{m_{ij}} = e \right\rangle$$



Coxeter groups realized as geometric reflection groups  
in Laplacians with delta ridge potentials

$$H = -\Delta_{\vec{x}} + \sum_{\xi} c_{\xi} \delta(\vec{n}_{\xi} \vec{x} + s_{\xi})$$

Betha ansatz solvable!

# 8. Permutation symmetry

Identical vs. indistinguishable

# Three main symmetrization methods for indistinguishable particles

- Calculate for distinguishable, then symmetrize
  - Space and spin separately, then combine
  - Or space and spin simultaneously

$$|\alpha\alpha\beta\rangle \xrightarrow{P_{[3]}} \frac{1}{\sqrt{3}} (|\alpha\alpha\beta\rangle + |\alpha\beta\alpha\rangle + |\beta\alpha\alpha\rangle)$$

$$|\alpha\alpha\beta\rangle \xrightarrow{P_{[2]}P_{[21]}} \frac{1}{\sqrt{6}} (|\alpha\alpha\beta\rangle + |\alpha\beta\alpha\rangle - 2|\beta\alpha\alpha\rangle)$$

$$|\alpha\alpha\beta\rangle \xrightarrow{P_{[1^2]}P_{[21]}} \frac{1}{\sqrt{2}} (|\alpha\alpha\beta\rangle - |\alpha\beta\alpha\rangle)$$

- Calculate using symmetrized second quantization creation/annihilation operators

- Take quotient of configuration space, do quantum theory on that

$$\mathbb{R}^{Nd} \rightarrow \frac{\mathbb{R}^{Nd}}{S_N}$$

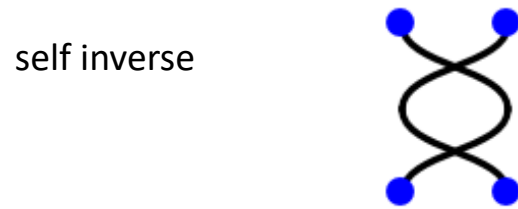
$$S_3$$

Key math concepts: permutation groups, normal subgroup, quotient group, point groups, indistinguishable vs. identical

# Symmetric group as strand diagrams

pairwise exchanges are generators:

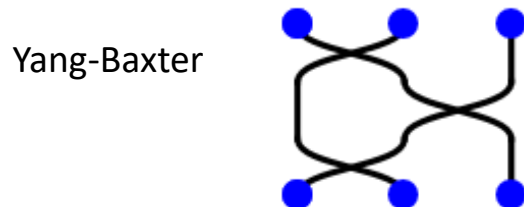
$$s_1, s_2, \dots, s_{N-1} \rightarrow S_N$$



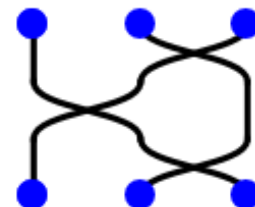
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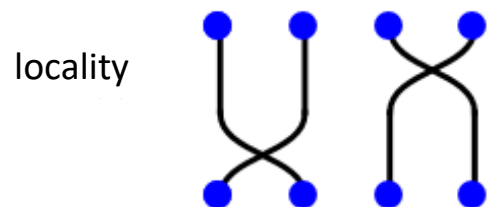
$$s_i^2 = 1$$



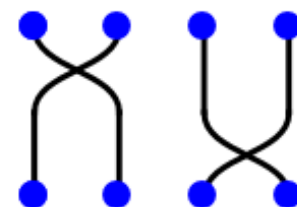
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$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$$



=



$$s_i s_j = s_j s_i \text{ when } j \geq i + 2$$

exchanges and permutations are the same

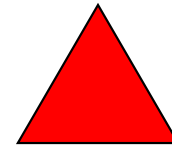
# Symmetric groups

Objects	Elements	Conjugacy Classes
2	2	2
3	6	3
4	24	5
5	120	7

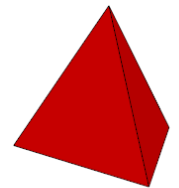
$$p_{12} = (1)(2) \quad p_{21} = (12)$$



$$p_{123} = (1)(2)(3) \quad p_{213} = (12) \quad p_{231} = (123)$$



$$p_{1234} = ( ) \quad p_{2134} = (12) \quad p_{2314} = (123) \quad p_{2143} = (12)(34) \quad p_{2341} = (1234)$$



One irreducible representation for every conjugacy class

# Standard Young Tableaux

- Young diagram:
  - There are  $N$  boxes in rows and columns.
  - Upper left justified
  - Each row has the same or fewer number of columns as row above.
  - Unique correspondence between partition of  $N$  and Young diagram
- Standard Young tableau:
  - Fill boxes with all numbers 1 through  $N$  used only once.
  - Numbers must increase to the right and to the bottom.
- Conjugate partitions :
  - Conjugate diagrams have rows and columns reversed
  - Conjugate diagrams have same number of standard Young tableaux

# Symmetric Group Irreps

Diagram illustrating two Young diagrams for the symmetric group  $S_2$ :

- Diagram 1: A horizontal row of two boxes containing 1 and 2, labeled  $[2]$ .
- Diagram 2: A vertical column of two boxes containing 1 and 2, labeled  $[1^2]$ .



Diagram illustrating four Young diagrams for the symmetric group  $S_3$ :




- Diagram 1: A horizontal row of three boxes containing 1, 2, and 3, labeled  $[3]$ .
- Diagram 2: A vertical column of two boxes containing 1 and 2, with a box containing 3 below the 2, labeled  $[21]$ .
- Diagram 3: A vertical column of two boxes containing 1 and 2, with a box containing 3 to the right of the 2, labeled  $[21]$ .
- Diagram 4: A vertical column of three boxes containing 1, 2, and 3, labeled  $[1^3]$ .

Diagram illustrating seven Young diagrams for the symmetric group  $S_4$ :

- Diagram 1: A horizontal row of four boxes containing 1, 2, 3, and 4, labeled  $[4]$ .
- Diagram 2: A vertical column of three boxes containing 1, 2, and 3, with a box containing 4 below the 3, labeled  $[31]$ .
- Diagram 3: A vertical column of two boxes containing 1 and 2, with boxes containing 3 and 4 to the right of the 2, labeled  $[2^2]$ .
- Diagram 4: A vertical column of two boxes containing 1 and 2, with a box containing 3 below the 2, and a box containing 4 to the right of the 3, labeled  $[21^2]$ .
- Diagram 5: A vertical column of two boxes containing 1 and 2, with a box containing 3 to the right of the 2, and a box containing 4 to the right of the 3, labeled  $[21^2]$ .
- Diagram 6: A vertical column of four boxes containing 1, 2, 3, and 4, labeled  $[1^4]$ .

# Character tables

$S_2$	$(1^2)$	$(2)$	
$[2]$	1	1	
$[1^2]$	1	-1	

$S_3$	$(1^3)_1$	$(12)_3$	$(3)_2$	
$[3]$	1	1	1	
$[21]$	2	0	-1	
$[1^3]$	1	-1	1	

# Irrep and character facts

- Dimensions of irreps  $\sum_{\mu} n_{\mu}^2 = [G]$

- Orthogonality of irrep characters

$$\sum_i [g_i] \chi^{(\mu)*}(g_i) \chi^{(\nu)}(g_i) = [G] \delta_{\mu\nu}$$

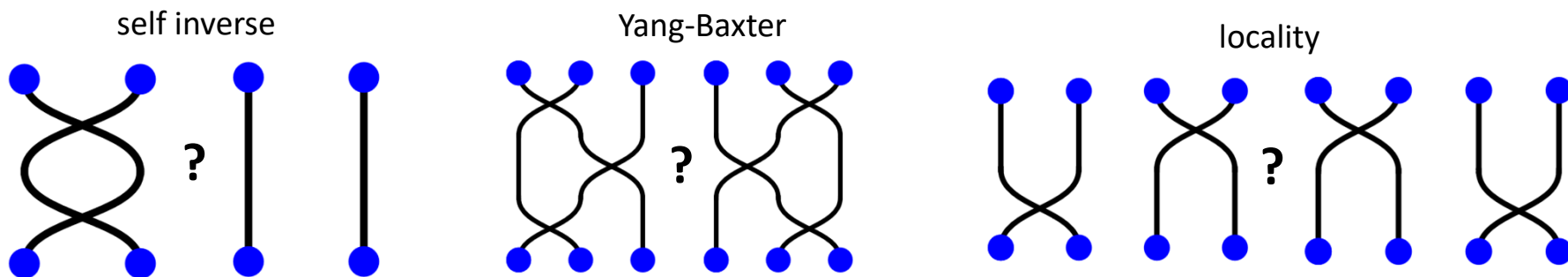
- Useful special case  $\sum_i [g_i] \left| \chi^{(\mu)}(g_i) \right|^2 = [G]$

$S_4$	$(1^4)$	$(1^2 2)_6$	$(2^2)$ <span style="background-color: #4a90e2; color: white; padding: 2px;">A</span>	$(13)_8$	$(4)_6$
$[4]$	1	1	1	1	1
$[31]$	3	1	-1	0	<span style="background-color: #4a90e2; color: white; padding: 2px;">D</span>
$[2^2]$	2	0	<span style="background-color: #4a90e2; color: white; padding: 2px;">C</span>	-1	0
$[21^2]$	<span style="background-color: #4a90e2; color: white; padding: 2px;">B</span>	-1	-1	0	1
$[1^4]$	1	-1	1	1	-1

$S_5$	$(1^5)$	$(1^3 2)_{10}$	$(12^2)_{15}$	$(1^2 3)_{20}$	$(23)_{20}$	$(14)_{30}$	$(5)_{24}$
[5]	1	1	1	1	1	1	1
[41]	4	2	0	1	-1	0	-1
[32]	5	1	1	-1	1	-1	0
[31 <sup>2</sup> ]	6	0	-2	0	0	0	1
[2 <sup>2</sup> 1]	5	-1	1	-1	-1	1	0
[21 <sup>3</sup> ]	4	-2	0	1	1	0	-1
[1 <sup>5</sup> ]	1	-1	1	1	-1	-1	1

Five is different

# Break the generating relations to get strand groups



Symmetric group



Braid group



Traid group



# Applications

- Particle permutation symmetry and exchange statistics
- State permutation symmetry
- Non-interacting particles
- Ordering permutation symmetry

# Wreath Products



What is the group of symmetry transformations?

permutations

$$S_3$$

reflections

$$D_1^{\times 3}$$

permutations and reflections don't commute

semidirect  
product

$$S_3 \rtimes D_1^{\times 3}$$



$$D_1 \wr S_3$$

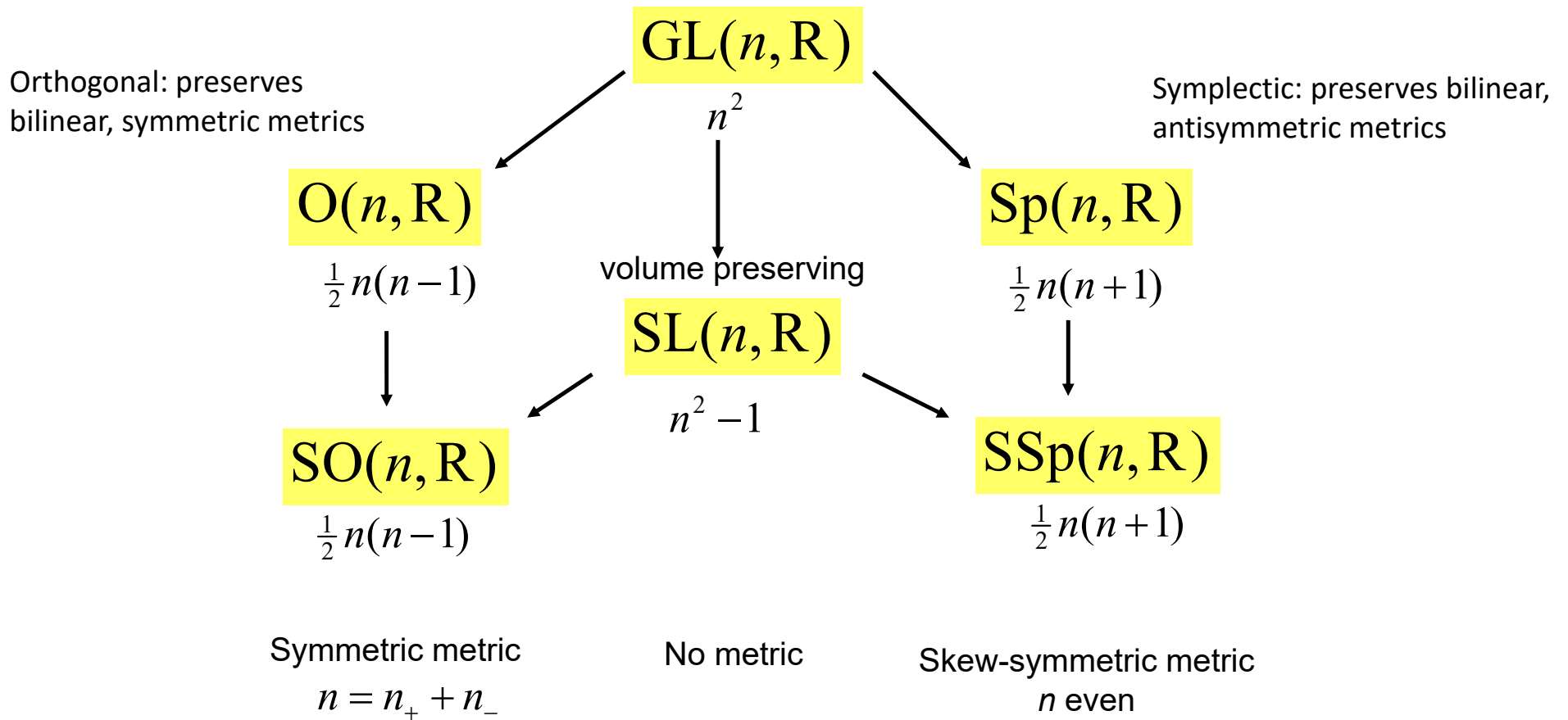
# Non-interacting identical quantum systems

$$\mathbf{T}_t \int \mathbf{S}_N$$

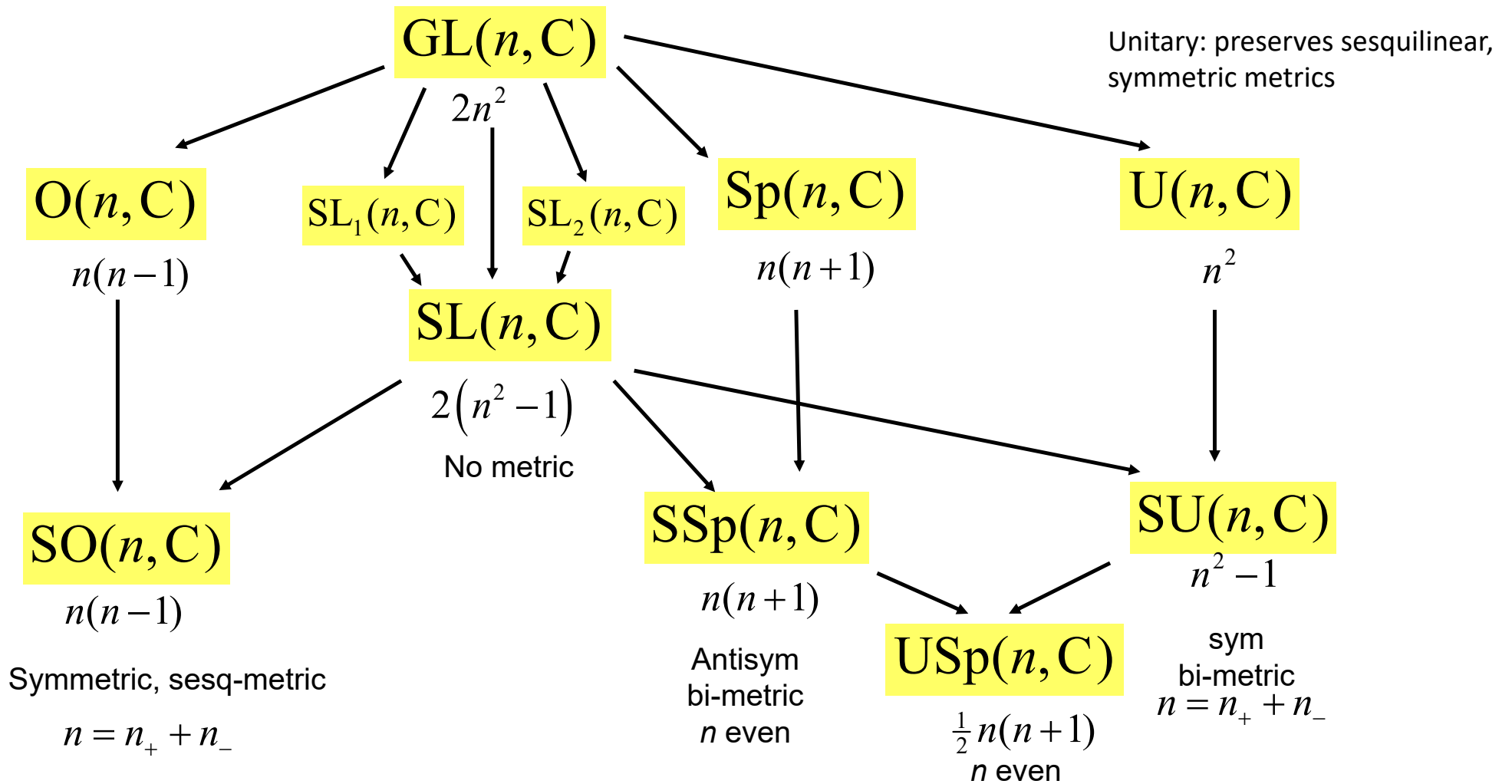
# 9. Unitary symmetry

More than spin!

# Interlude: Matrix groups



Adapted from Lie Groups, Lie Algebras, and Some Applications, R. Gilmore, (Wiley, New York, 1974)

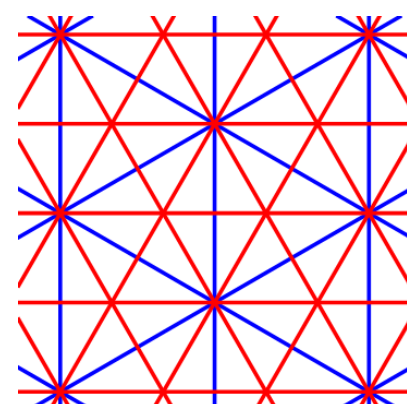
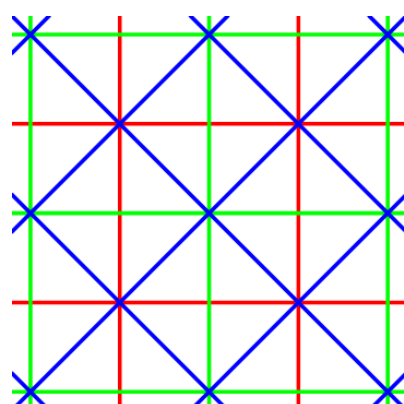
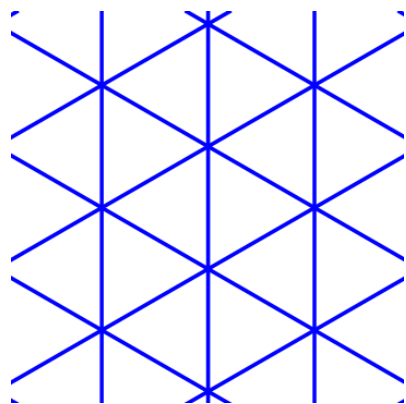
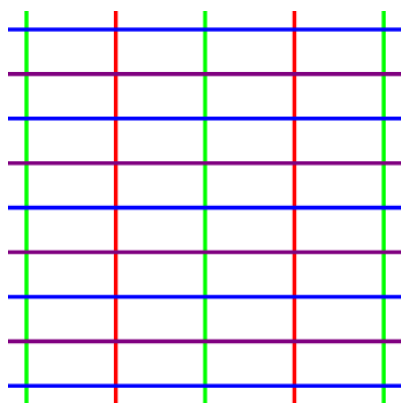


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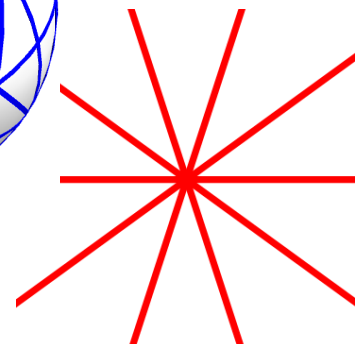
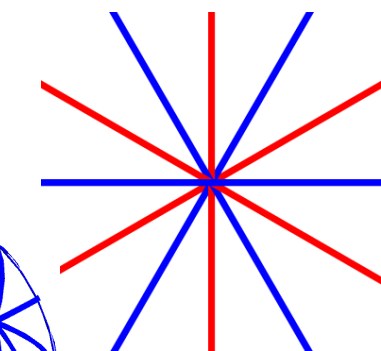
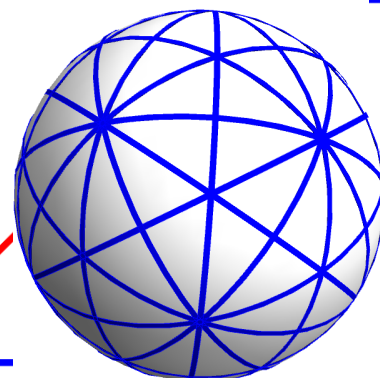
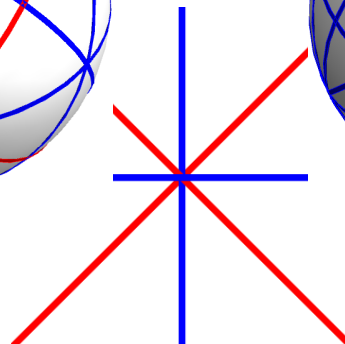
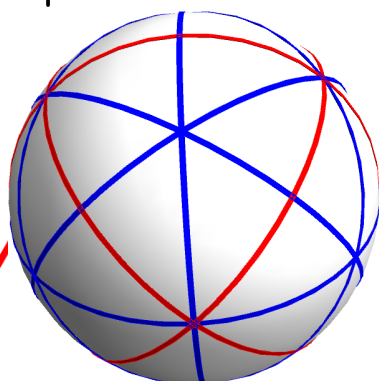
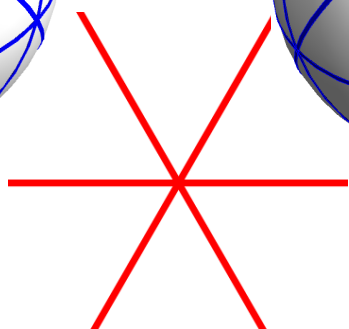
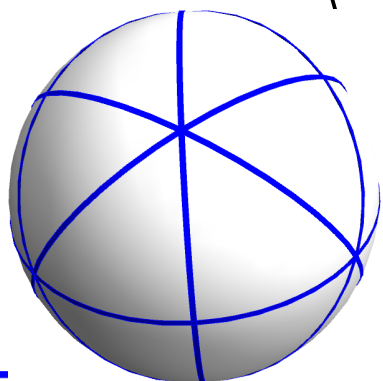
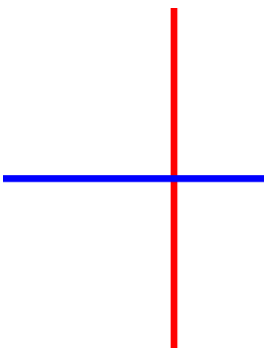
# Compact Lie Groups from Simple Lie Algebras

- A series       $A_1, A_2, A_3, \dots$        $SU(r + 1)$
- B series       $B_2, B_3, B_4, \dots$        $SO(2r + 1)$
- C series       $C_3, C_4, C_5, \dots$        $Sp(2r)$
- D series       $D_4, D_5, D_6, \dots$        $SO(2r)$
- Exceptional Lie groups

$$G_2, F_4, E_6, E_7, E_8$$



$$\left\langle \sigma_1, \dots, \sigma_r \mid \sigma_i^2 = (\sigma_i \sigma_j)^{m_{ij}} = e \right\rangle$$



# Universal Covering Group

- UCG  $\tilde{G}$  of Lie group  $G$  is a unique, simply connected Lie group with a homomorphism  $\tilde{G} \rightarrow G$  and isomorphic Lie algebras.

$SU(2)$  is UCG of  $SO(3)$

$$SO(3) \simeq SU(2)/Z_2$$

- UCG of  $SO(n)$  sometimes called  $Spin(n)$

# Harmonic oscillators

- Kinematic symmetry of N-dimensional isotropic harmonic oscillator

$$H = \frac{1}{2} \sum_{i=1}^N (p_i^2 + q_i^2) = \sum_{i=1}^N \left( a_i^\dagger a_i + \frac{1}{2} \right) = \vec{\mathbf{a}}^\dagger \vec{\mathbf{a}} + N / 2$$

$$O(2N) \cap Sp(2N, \mathbf{R}) = U(N)$$

Rotations in  
phase space

Canonical linear  
transformations

- Lie algebra of quadratic operators

$$L_{ij} = Q_i P_j - Q_j P_i = -i(a_i^\dagger a_j - a_j^\dagger a_i)$$

$$D_{ij} = Q_i Q_j + P_i P_j = -i(a_i^\dagger a_j - a_j^\dagger a_i) \quad C_{ij} = N_i - N_j = (a_i^\dagger a_i - a_j^\dagger a_j)$$

# 10. Galilean symmetry

Non-relativistic relativity

# Galilean Symmetry

- Translations in space

$$\mathbf{x}' = \mathbf{x} + \mathbf{a} \implies \hat{\mathbf{P}} \implies \hat{U}(\mathbf{a}) = \exp(-i\mathbf{a} \cdot \hat{\mathbf{P}})$$

- Translations in time

$$t' = t + b \implies \hat{H} \implies \hat{U}(b) = \exp(-ib\hat{H})$$

- Rotations

$$\mathbf{x}' = \underline{R}\mathbf{x} \implies \hat{\mathbf{J}} \implies \hat{U}(\underline{R}) = \exp(-i\boldsymbol{\theta} \cdot \hat{\mathbf{J}})$$

- Galilean boosts

$$\mathbf{x}' = \mathbf{x} + \mathbf{v}t \implies \hat{\mathbf{K}} = m\hat{\mathbf{X}} \implies \begin{aligned} \hat{U}(\mathbf{v}) &= \exp(-i\mathbf{v} \cdot \hat{\mathbf{K}}) \\ &= \exp(-im\mathbf{v} \cdot \hat{\mathbf{X}}) \end{aligned}$$

Kinematic  
symmetries of  
free-space  
Hamiltonian

Dynamic  
symmetries of  
free-space  
Hamiltonian

# Complete Set of Commuting Observables for Quantum One-Body Problem

- For one-body problem in free space

$$\{\hat{\mathbf{X}}, \hat{S}_z\} \quad \{\hat{\mathbf{P}}, \hat{S}_z\} \quad \{\hat{H}_0, \hat{\mathbf{L}}^2, \hat{L}_z, \hat{S}_z\} \quad \{\hat{H}_0, \hat{\mathbf{J}}^2, \hat{\mathbf{L}}^2, \hat{J}_z\}$$

- For spherical symmetry

$$[\hat{H}, \hat{\mathbf{J}}] = 0 \quad \{\hat{H}, \hat{\mathbf{J}}^2, \hat{\mathbf{L}}^2, \hat{J}_z\}$$

- For central potential

$$[\hat{H}, \hat{\mathbf{L}}] = [\hat{H}, \hat{\mathbf{S}}] = 0 \quad \{\hat{H}, \hat{\mathbf{L}}^2, \hat{L}_z, \hat{S}_z\}$$

## Next 8 best symmetries

- Symplectic symmetry
- Conformal symmetry
- Exchange symmetries
- Lorentz symmetry
- Poincare symmetry
- Gauge symmetry
- Schrodinger symmetries
- Oscillator symmetries

$$SL(2, \mathbb{R}) \sim Sp(2, \mathbb{R}) \sim SO(2, 1)$$

Key math concepts:

Key physical applications: conformal symmetry, squeezed states and two-mode interferometers


# Key applications of group theory to symmetry in quantum theory

1. Algebraic solvability and harmonic analysis
2. Classification of possible symmetries
3. Selection rules
4. Spectroscopic labels and degeneracy
5. Numerical efficiency and symmetry-adapted bases
6. Invariance, invariants, and covariance
7. Indistinguishable particles
8. Symmetry breaking and phase transitions
9. Integrability and chaos
10. Topological control


# A Simple Few-Body Model

$$H = \sum_{i=1}^N \left( \frac{1}{2m} p_i^2 + V(x_i) \right) + g \sum_{\langle i,j \rangle} \delta(x_i - x_j)$$


particles moving in  
one dimension



all particles feel same  
trapping potential



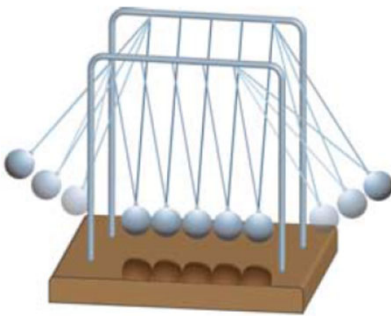
contact (zero-range)  
pairwise interactions



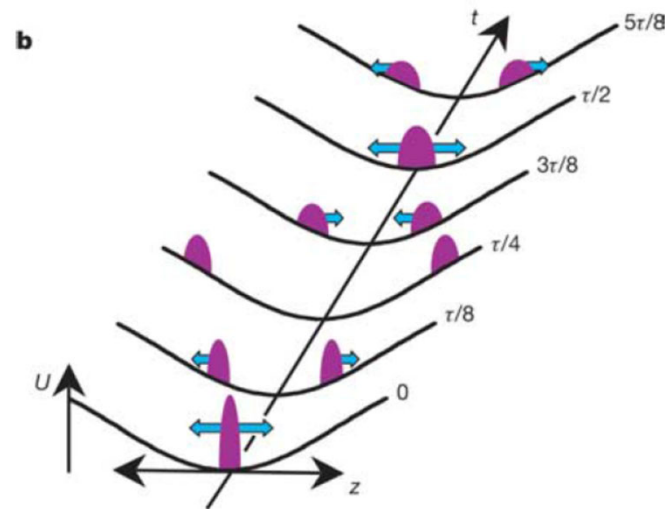
# A quantum Newton's cradle

Toshiya Kinoshita, Trevor Wenger and David S. Weiss  
Nature 440, 900-903 (13 April 2006)

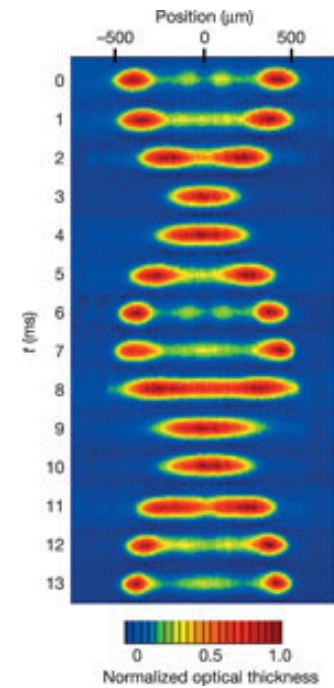
classical integrable  
few-body system



one-dimensional BEC tuned  
near unitary limit of contact  
interactions

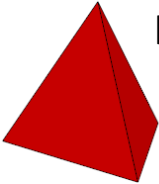


delayed thermalization



# Embellish the Model

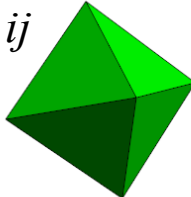
indistinguishable particles

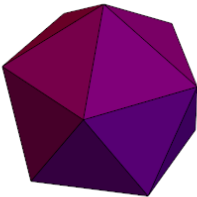


$$H = \sum_{i=1}^N \left( \frac{1}{2m} p_i^2 + V(x_i) \right) + g \sum_{\langle i,j \rangle} \delta(x_i - x_j) \left( m_i \dot{x}_i^2 + V_i(x_i) \right) + g_{ij}$$

particle have multiple components like spin or internal structure

particles, traps, or interactions not homogenous





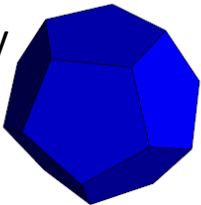
$$+ \sum_{\langle i,j,k \rangle} W(x_i, x_j, x_k) + \dots$$

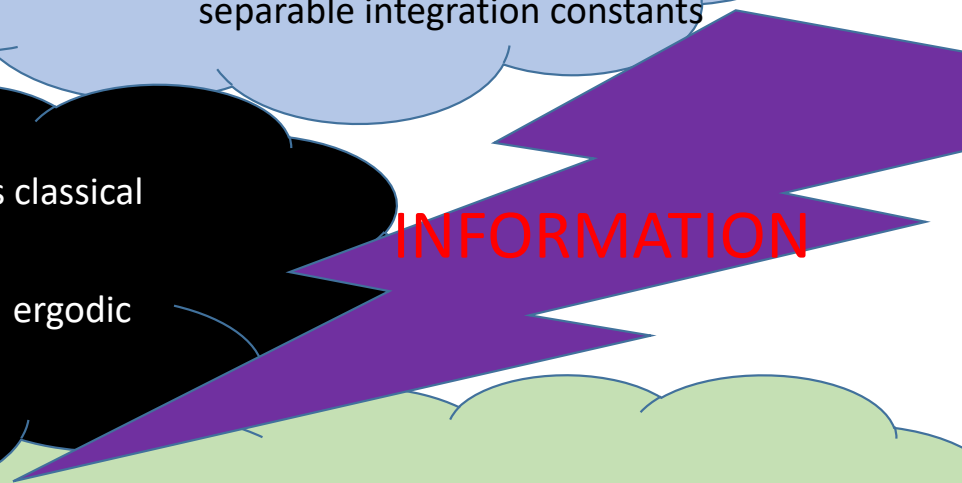
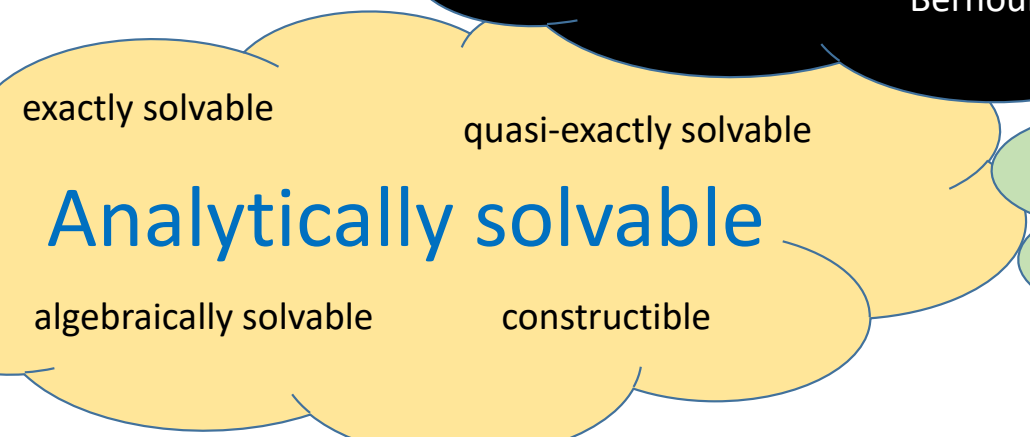
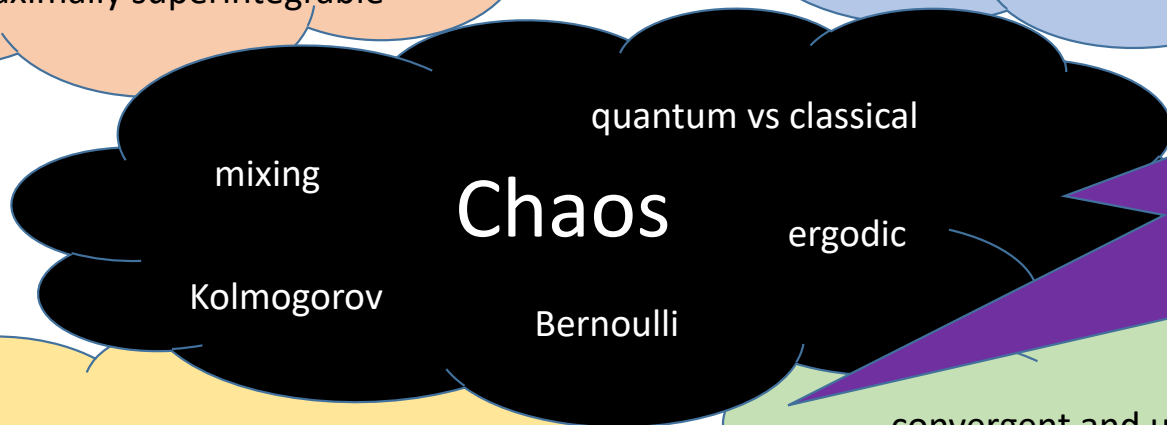
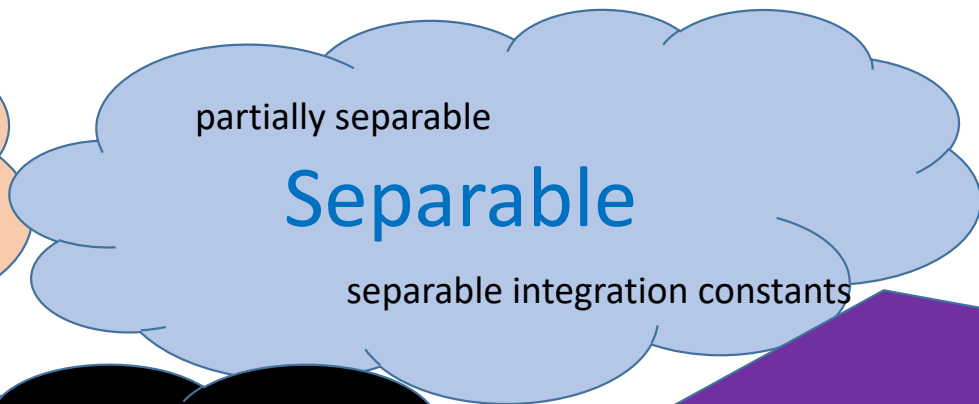
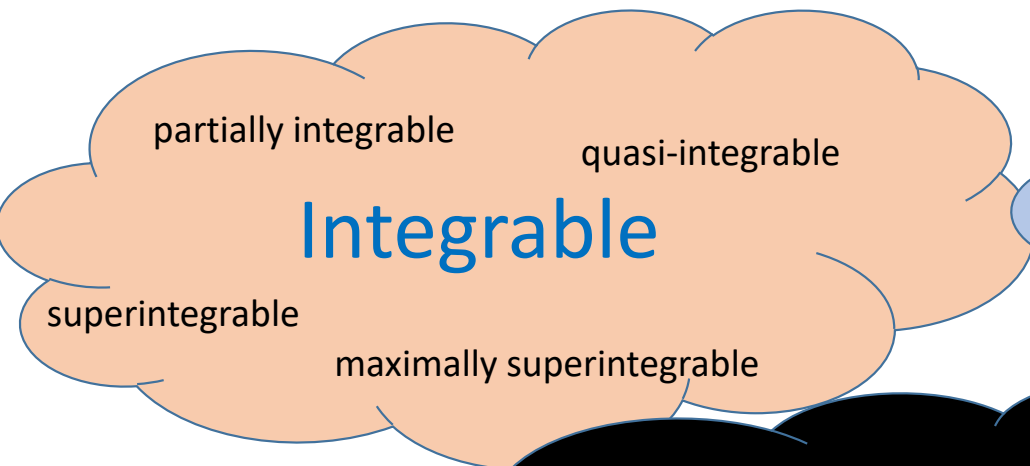
intrinsic few-body interactions

time-dependent, more dimensions, open system issues like noise and loss,...

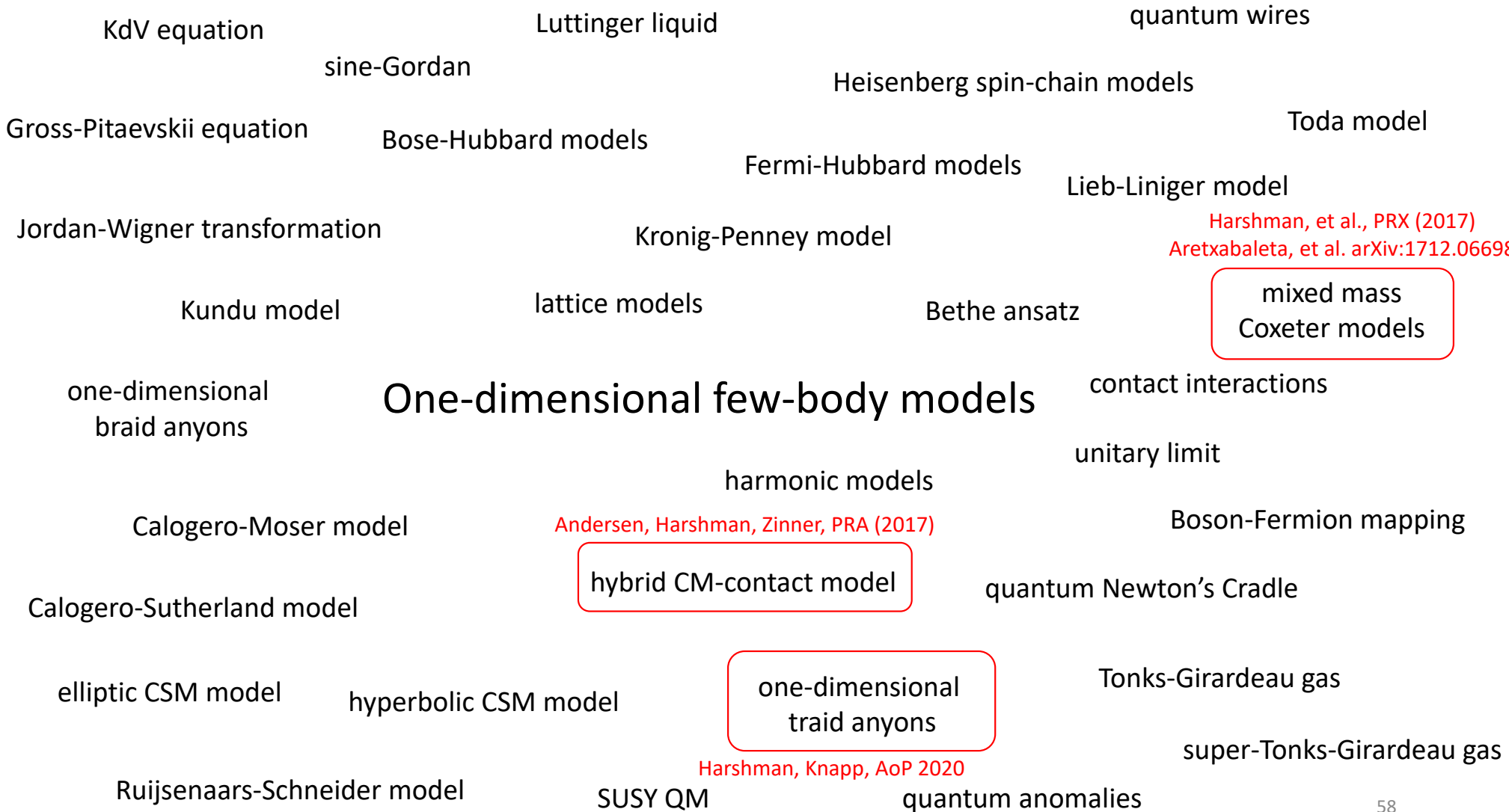
$$U_{\sigma\sigma'}(|x_i - x_j|)$$

more general two-body interactions





# One-dimensional few-body models



# Degeneracy Theorem

- Every energy eigenspace must carry a representation of a symmetry group of the Hamiltonian.
  - Irreps of subgroups are good labels for degenerate energy eigenspaces
- Special case: all the energy eigenspaces carry irreducible representations of the symmetry group of the Hamiltonian.
  - Then we the *kinematic symmetries* of the Hamiltonian are sufficient to explain the degeneracies.
- Unexplained degeneracies?
  - We could be missing a kinematic symmetry...keep looking!
  - Or...there could be dynamic symmetries.
  - Or there could be accidental symmetries. ???

## Three-dimensional isotropic harmonic oscillator

$$\hat{H} = \sum_{i=1}^3 \left( -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x_i^2} + \frac{1}{2} m \omega^2 x_i^2 \right) = \frac{\hbar \omega}{2} \sum_{i=1}^3 \left( -\frac{\partial^2}{\partial q_i^2} + q_i^2 \right)$$

- U(3) symmetry “explains” degeneracy

$$E = \hbar \omega (X + 3/2) \quad d(X) = (X + 2)(X + 1) / 2$$

- Lots of choices for CSCOs because of U(3) symmetry, e.g.

$$\{H, L^2, L_3\}, \{N_1, N_2, N_3\}, \{H, N_3, L_3\}$$

spherical

cartesian

cylindrical

$$O(3)$$

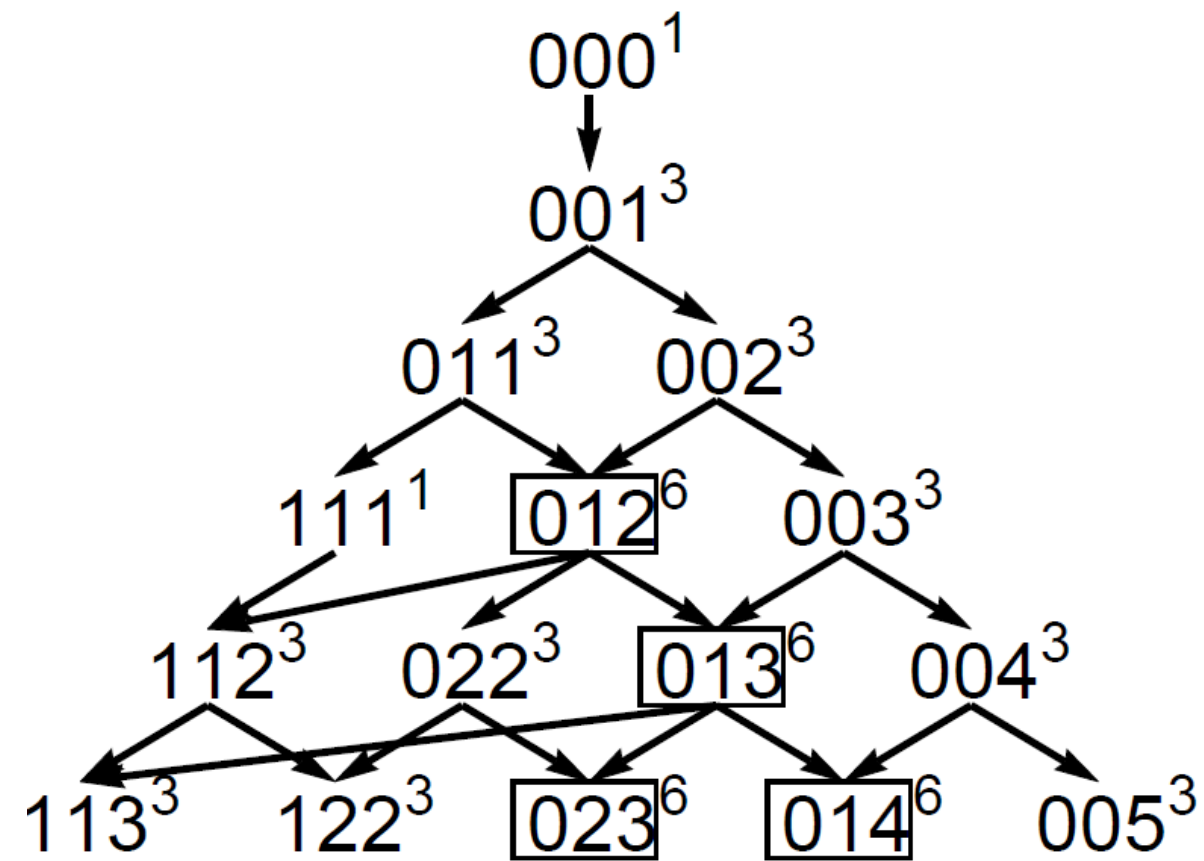
$$U(1) \times U(1) \times U(1)$$

$$U(1) \times O(2)$$

anharmonic perturbation

anisotropic perturbation

three 1-D particle interactions



X	TOTAL	BOSON	FERMION
0	1	1	0
1	3	1	0
2	6	2	0
3	10	3	1
4	15	4	1
5	21	5	2

# Three one-dimensional interacting particles in an harmonic trap

- Native particle coordinates

$$\hat{H} = \hat{H}_0 + \hat{V} \quad \hat{H}_0 = \frac{\hbar\omega}{2} \sum_{i=1}^3 \left( -\frac{\partial^2}{\partial q_i^2} + q_i^2 \right) \quad \hat{V} = g \sum_{\langle i,j \rangle} \delta(q_i - q_j)$$

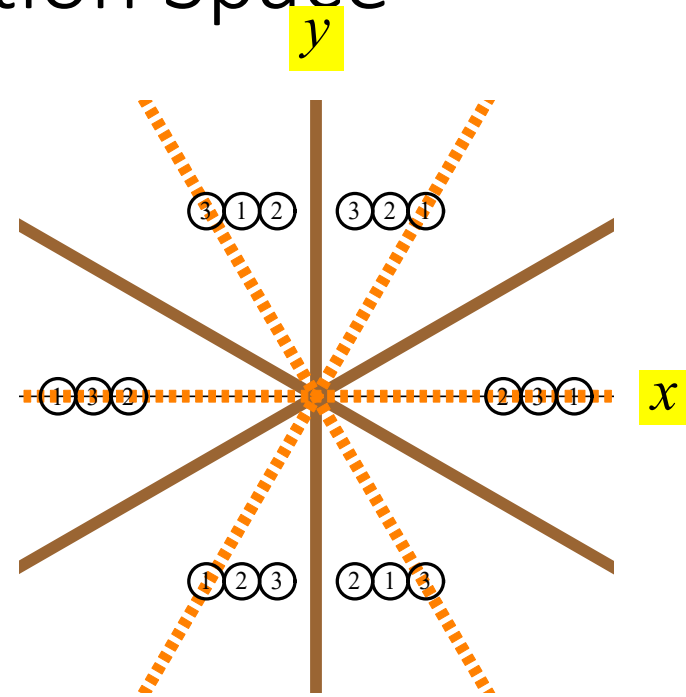
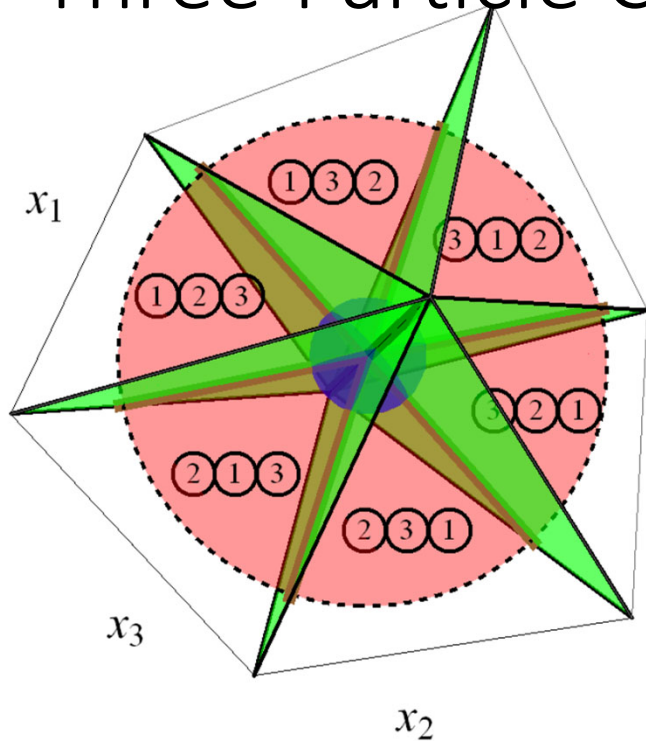
- Jacobi coordinates

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} \quad \begin{aligned} \rho^2 &= x^2 + y^2 \\ \tan \varphi &= y / x \end{aligned}$$

- COM and relative separation  $\hat{H} = \hat{H}_z + \hat{H}_{rel} \quad \hat{H}_z = \frac{\hbar\omega}{2} \sum_{i=1}^3 \left( -\frac{\partial^2}{\partial z^2} + z^2 \right)$

$$\hat{H}_{rel} = \frac{\hbar\omega}{2} \sum_{i=1}^3 \left\{ -\frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial}{\partial \rho} \right) - \frac{1}{\rho^2} \frac{\partial^2}{\partial \varphi^2} + \rho^2 \right\} + \frac{g}{\sqrt{2}\rho} \sum_{i=0}^5 \delta\left(\varphi - \frac{\pi(2i+1)}{6}\right)$$

# Three-Particle Configuration Space



Configuration space  
symmetry group:

$$S_3 \times O(1) \times O(1) \sim D_{6h}$$

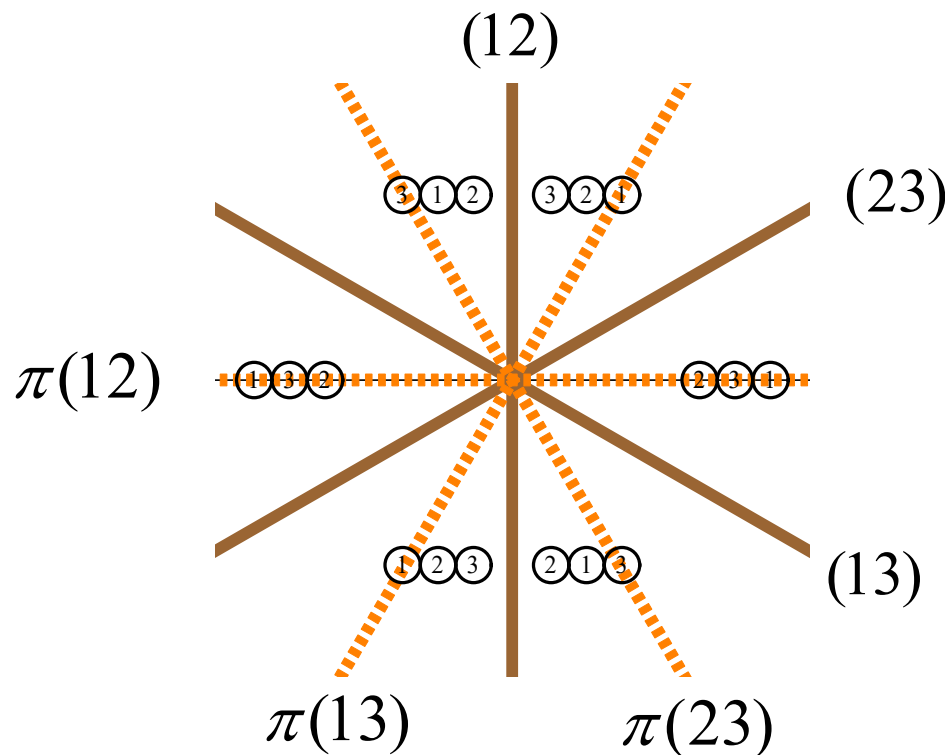
Maximal kinematic  
symmetry group:

$$S_3 \times O(1) \times U(1) \times T_t$$

Relative configuration  
space symmetry group:

$$S_3 \times O(1) \square C_{6v}$$

# Three-Particle Configuration Space



Symmetry group:

$$S_3 \times O(1) \sim C_{6v}$$

Relative inversion:

$$\pi \rightarrow C_2$$

Reflections:

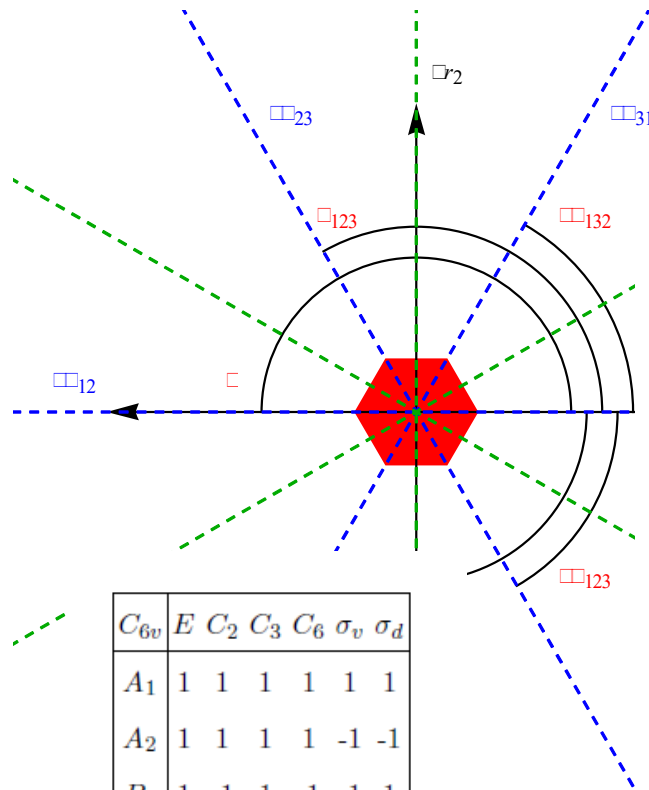
$$(12), (13), (23)$$

$$\pi(12), \pi(13), \pi(23)$$

Rotations:

$$(123), (132) \rightarrow 2C_3$$

$$\pi(123), \pi(132) \rightarrow 2C_6$$



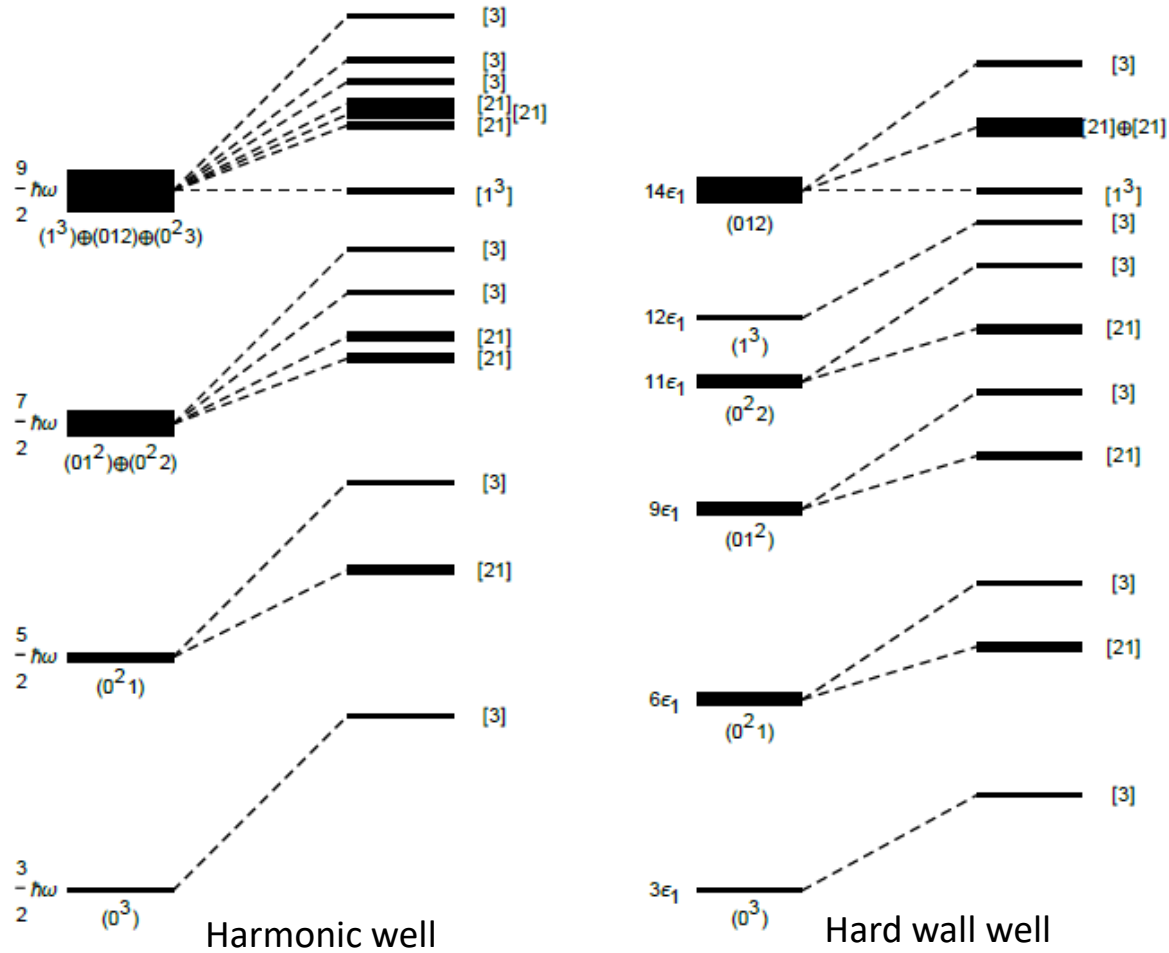
$C_{6v}$	$E$	$C_2$	$C_3$	$C_6$	$\sigma_v$	$\sigma_d$
$A_1$	1	1	1	1	1	1
$A_2$	1	1	1	1	-1	-1
$B_1$	1	-1	1	-1	-1	1
$B_2$	1	-1	1	-1	1	-1
$E_1$	2	-2	-1	1	0	0
$E_2$	2	2	-1	-1	0	0

$\mu$	$\pm$	$C_{6v}$	$C_{2v}$	Possibilities
0	N.A.	$A_1$	$A_1$	BBB, BBX, XYZ
1	+	$E_1$	$B_1$	FFX, XYZ
1	-	$E_1$	$B_2$	BBX, XYZ
2	+	$E_2$	$A_1$	BBX, XYZ
2	-	$E_2$	$A_2$	FFX, XYZ
3	+	$B_1$	$B_1$	FFF, FFX, XYZ
3	-	$B_2$	$B_2$	BBB, BBX, XYZ
4	+	$E_2$	$A_1$	BBX, XYZ
4	-	$E_2$	$A_2$	FFX, XYZ
5	+	$E_1$	$B_1$	FFX, XYZ
5	-	$E_1$	$B_2$	BBX, XYZ
6	+	$A_1$	$A_1$	BBB, BBX, XYZ
6	-	$A_2$	$A_2$	FFF, FFX, XYZ

$g \in C_{6v}$	$g \in S_3 \times Z_2$	$\varphi \rightarrow \varphi'$
$E$	$\hat{e}$	$\varphi$
$\sigma_v$	$\hat{\sigma}_{12}$	$-\varphi + \pi$
	$\hat{\sigma}_{23}$	$-\varphi + \frac{\pi}{3}$
	$\hat{\sigma}_{31}$	$-\varphi - \frac{\pi}{3}$
	$\hat{231}$	$\varphi - \frac{2\pi}{3}$
	$\hat{312}$	$\varphi + \frac{2\pi}{3}$
	$\hat{\pi}$	$\varphi + \pi$
	$\hat{\sigma}_{12}$	$-\varphi$
	$\hat{\sigma}_{23}$	$-\varphi - \frac{2\pi}{3}$
	$\hat{\sigma}_{31}$	$-\varphi + \frac{2\pi}{3}$
	$\hat{231}$	$\varphi + \frac{\pi}{3}$
	$\hat{312}$	$\varphi - \frac{\pi}{3}$

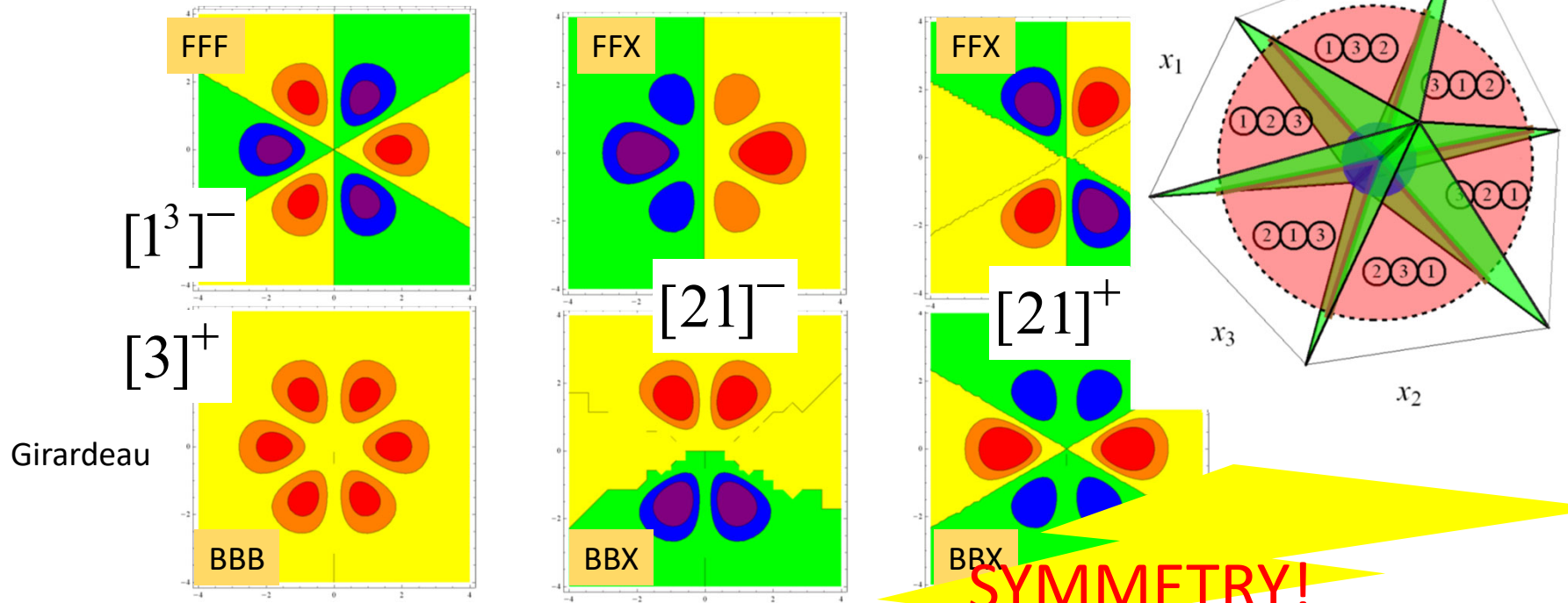
transformation designs

# Weak Perturbation



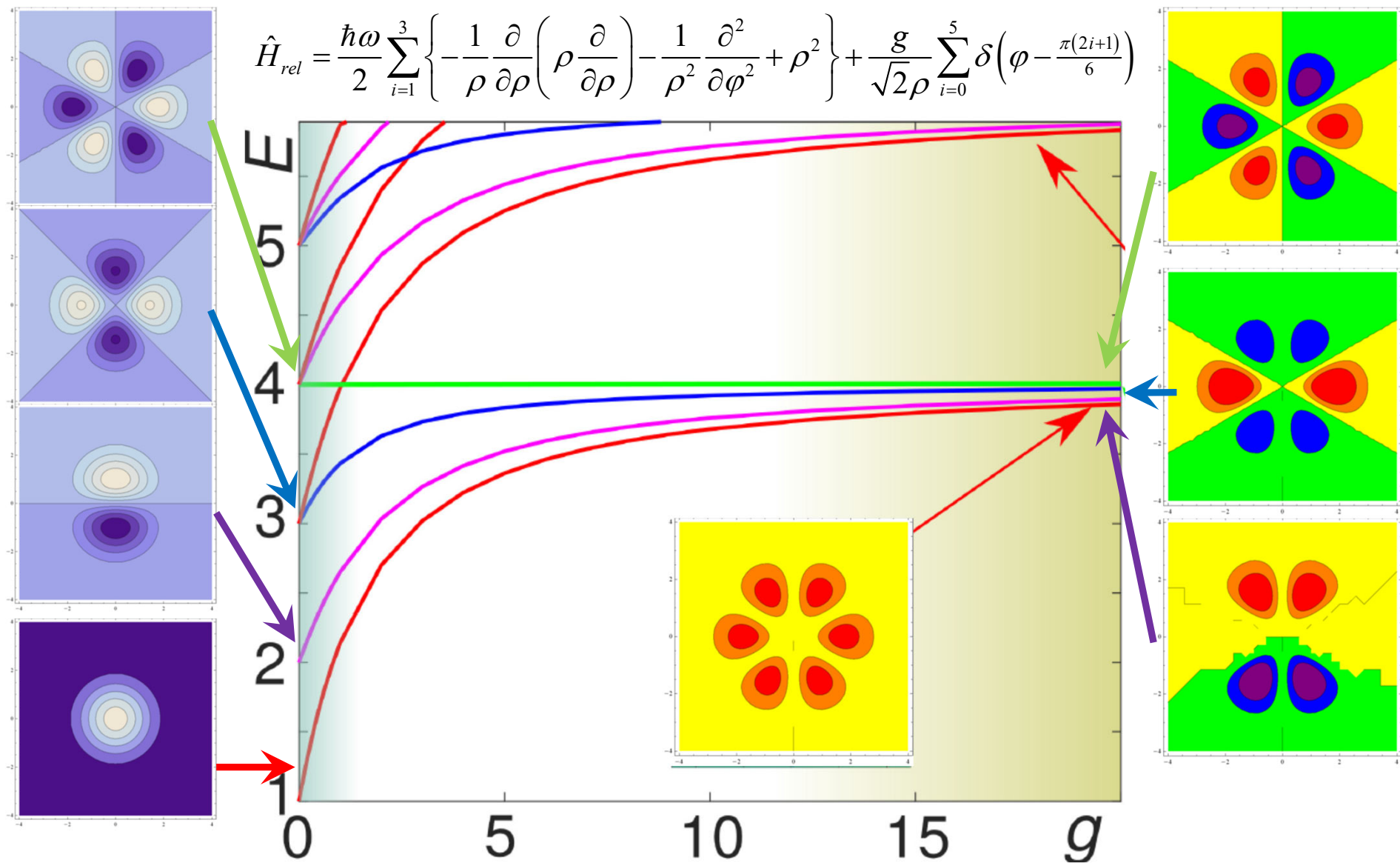
# Three Particles: Unitary Limit

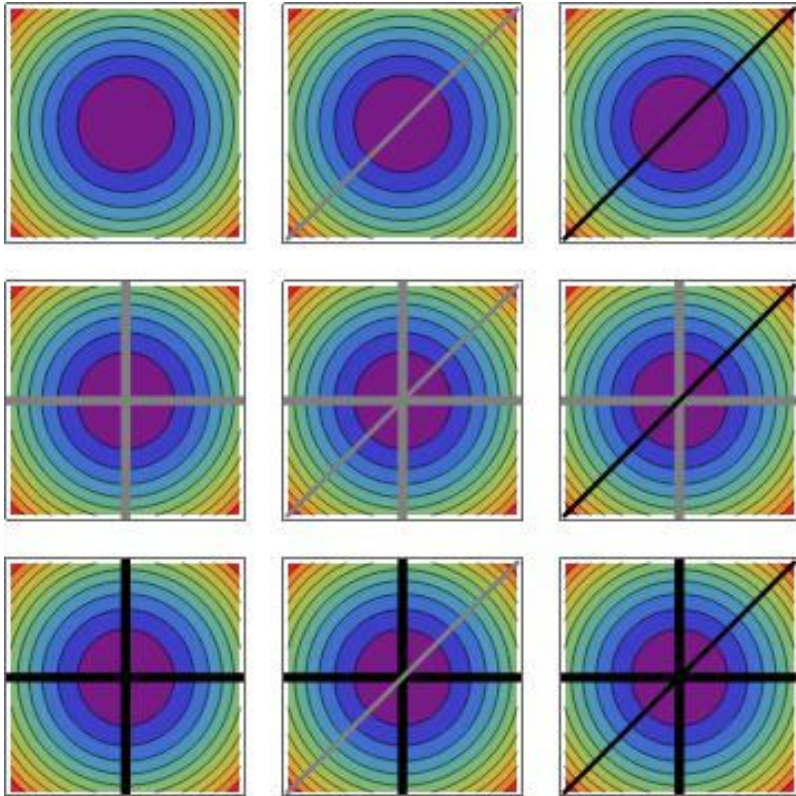
Starting from free fermionic solutions, one can build t



$E = \frac{9}{2}$  for all

$T_t \int S_{3!} \subset S_3 \times S_3$





<p>A</p> $O(2)$ $\infty$ $U(2)$	<p>S</p> $S_2 \times Z_2$ $4$ $S_2 \times U(1) \times T_t$	<p>A</p> $S_2 \wr Z_2$ $8$ $S_2 \wr (Z_2 \times T_t) \times U(1)$
<p>S</p> $S_2 \wr Z_2$ $8$ $S_2 \wr (Z_2 \times T_t)$	<p>N</p> $S_2 \times Z_2$ $4$ $S_2 \times Z_2 \times T_t$	<p>S</p> $S_2 \wr Z_2$ $8$ $S_2 \wr (Z_2 \times T_t)$
<p>A</p> $S_4 \wr Z_2$ $384$ $S_4 \wr (Z_2 \times T_t)$	<p>S</p> $(S_2 \wr Z_2)^{\times 2}$ $64$ $(S_2 \wr Z_2 \times T_t)^{\oplus 2}$	<p>A</p> $(S_2 \wr Z_2) \oplus S_4$ $192$ $(S_2 \wr Z_2 \times T_t) \oplus (S_4 \wr T_t)$

Where symmetry breaks, math ends and physics begins.

# Applications

- Particle permutation symmetry

# Normal Exchange Statistics:

particle permutations of identical particles given by symmetric group

$$p \in \mathcal{S}_N$$

- Bosons

$$\hat{U}(p)\psi(x_1, x_2, \dots, x_N) \equiv \psi(x_{p_1}, x_{p_2}, \dots, x_{p_N}) = \psi(x_1, x_2, \dots, x_N)$$

abelian

- Fermions

$$\hat{U}(p)\psi(x_1, x_2, \dots, x_N) = \begin{cases} \psi(x_1, x_2, \dots, x_N) & p \in \text{even} \\ -\psi(x_1, x_2, \dots, x_N) & p \in \text{odd} \end{cases}$$

- Parastatistics (useful for partially distinguishable identical particles)

$$\hat{U}(p)\psi_i(x_1, x_2, \dots, x_N) = \sum_j D_{ij}(p)\psi_j(x_1, x_2, \dots, x_N)$$

non-abelian

# Applications

- Particle permutation symmetry
- State permutation symmetry



# Applications

- Particle permutation symmetry
- State permutation symmetry
- Non-interacting particle symmetry

# Wreath Products



What is the group of symmetry transformations?

permutations

$$S_3$$

reflections

$$D_1^{\times 3}$$

permutations and reflections don't commute

semidirect  
product

$$S_3 \hat{a} D_1^{\times 3}$$



$$D_1 \wr S_3$$

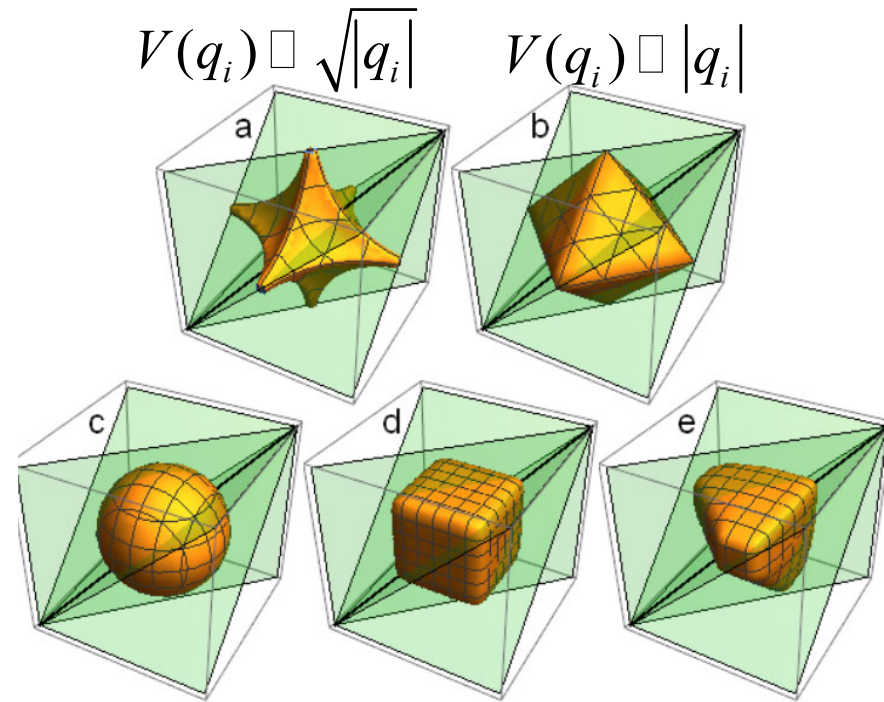
# Non-interacting identical quantum systems

$$\mathbf{T}_t \int \mathbf{S}_N$$

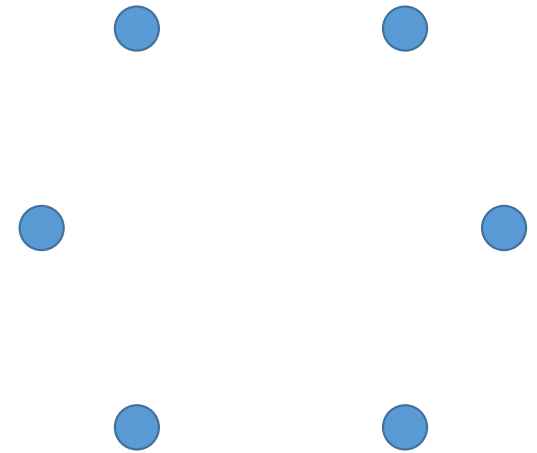
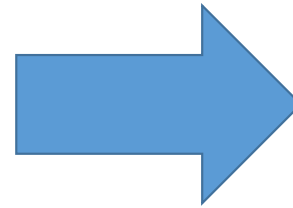
# Applications

- Particle permutation symmetry
- State permutation symmetry
- Non-interacting particles
- Ordering permutation symmetry

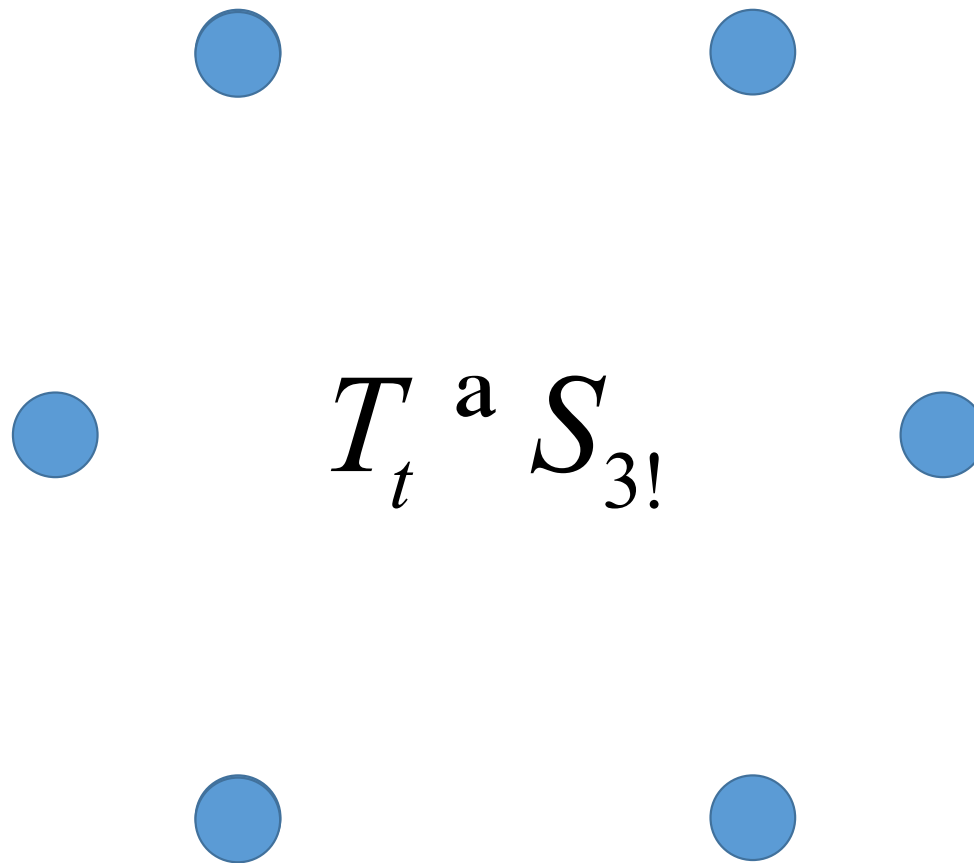
# Disconnected Ordering Sectors

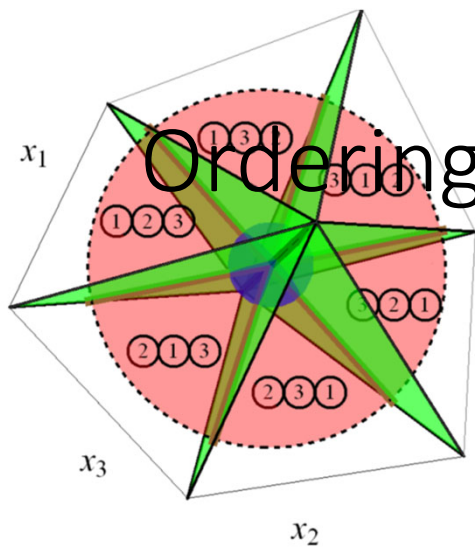


$$\begin{array}{l}
 V(q_i) \propto \sqrt{|q_i|} \\
 V(q_i) \propto |q_i| \\
 V(q_i) \propto q_i^2 \\
 V(q_i) \propto q_i^{10} \\
 V(q_i) \propto \begin{cases} q_i^{10} & q_i > 0 \\ |q_i| & q_i < 0 \end{cases}
 \end{array}$$

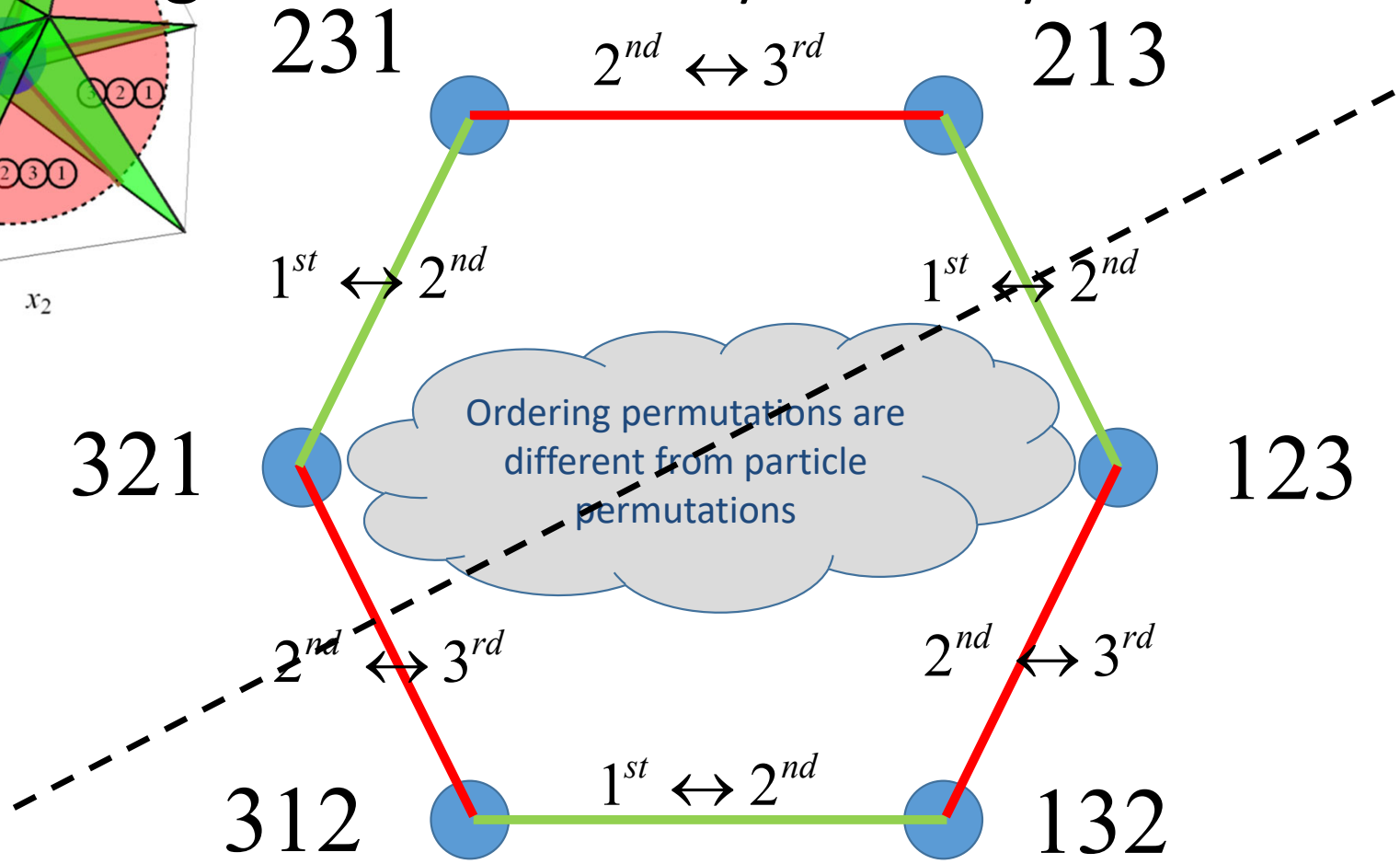


Unitary limit: Ordering permutation symmetry



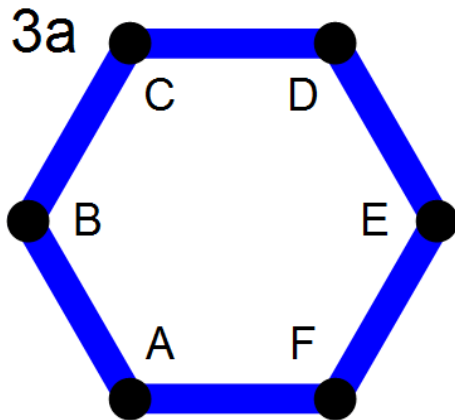


# Ordering Permutation Symmetry



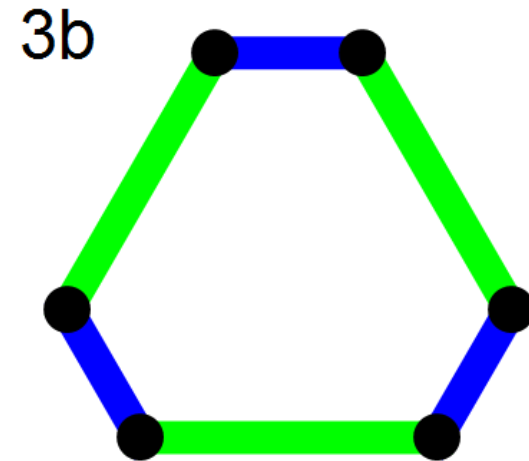
# Near-Unitary Limit

Symmetric Well



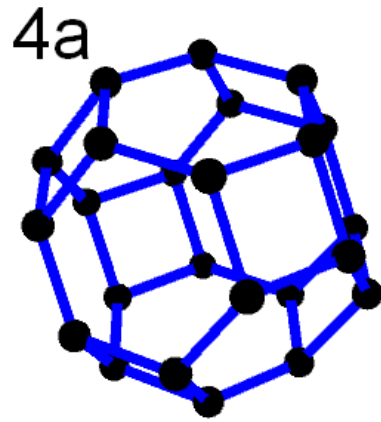
1-2 and 2-3 tunneling rates the same

Asymmetric Well

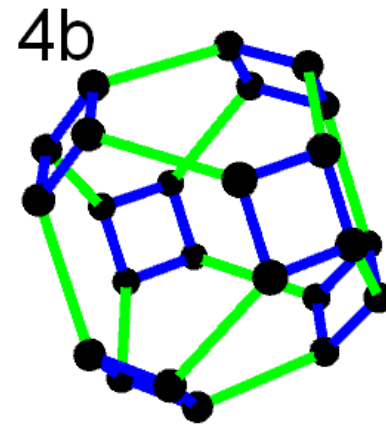


1-2 and 2-3 tunneling rates different

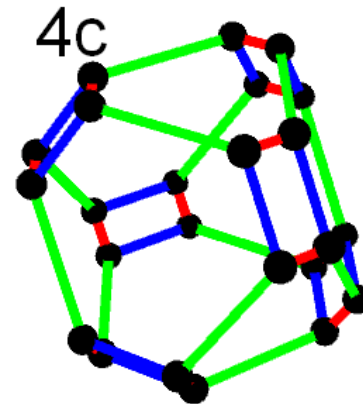
# Four particles



Square Well

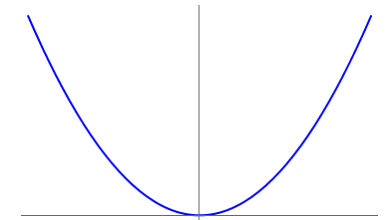
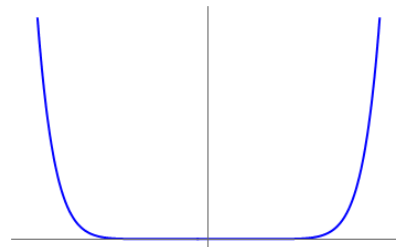
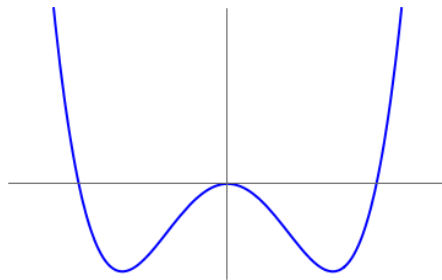
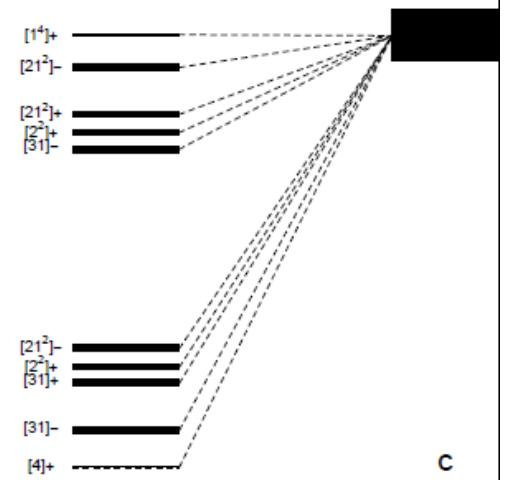
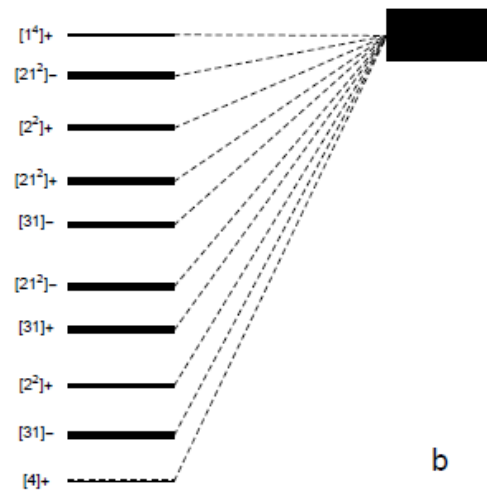
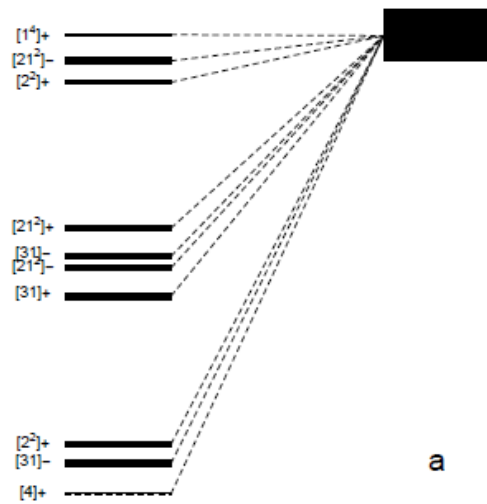
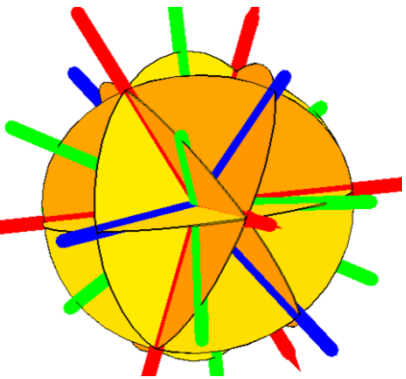


Symmetric Well



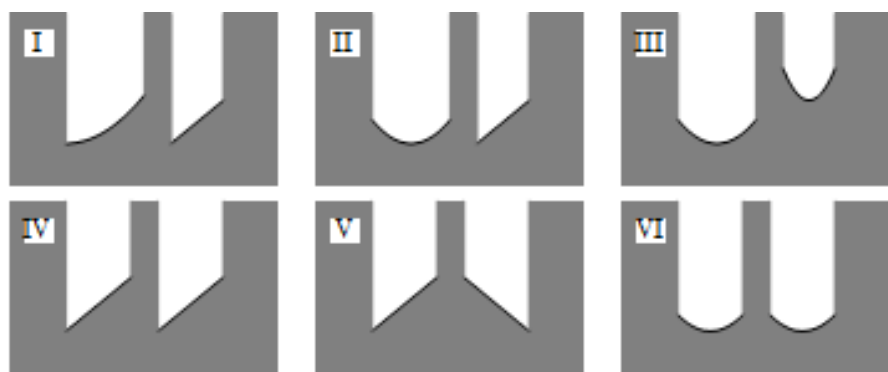
Asymmetric Well

# Two parameters for four particles in symmetric trap

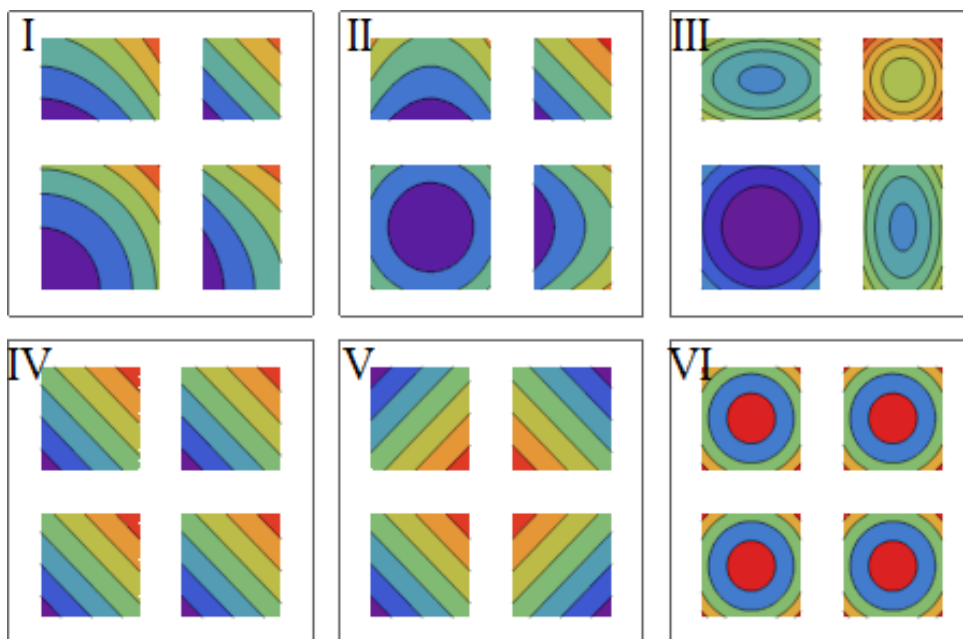


# Applications

- Particle permutation symmetry
- State permutation symmetry
- Non-interacting particles
- Ordering permutation symmetry
- Well permutation symmetry



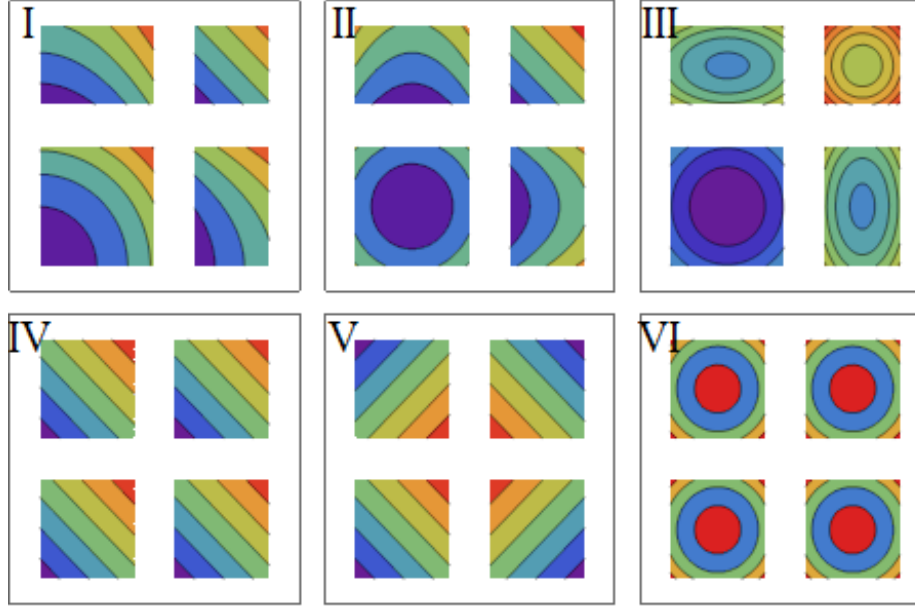
Type	$C_1^\infty$	$ C_1^\infty $	$C_1^r$	$ C_1^r $
I	E	1	E	1
II	$O(1)_a$	2	E	1
III	$O(1)_a \times O(1)_b$	4	E	1
IV	$W_2$	2	E	1
V	$W_2'$	2	$O(1)$	2
VI	$O(1) \wr W_2$	8	$O(1)$	2



$\gamma$	$K$
$\gamma = 0$	$(T_a \times \Pi_a) \int P_{2A} \times (T_{a+b} \times \Pi_a) \int W_{BD} \times T_b \int P_{2C}$
$\gamma \neq 0$	$\Pi_a \times P_{2A} \times (T_{a+b} \times \Pi_a) \int W_{BD} \times P_{2C}$
$\gamma \rightarrow \infty$	$(T_a \times \Pi_a) \int O_{2A} \times (T_{a+b} \times \Pi_a) \int W_{BD} \times T_b \int O_{2C}$

$\gamma$	$K$
$\gamma = 0$	$T_a \int P_{2A} \times T_{a+b} \int W_{BD} \times T_b \int P_{2C}$
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$\gamma \rightarrow \infty$	$T_a \int O_{2A} \times T_{a+b} \int W_{BD} \times T_b \int O_{2C}$

$\gamma$	$K$
$\gamma = 0$	$(T_a \times \Pi_a) \int P_{2A} \times (T_{a+b} \times \Pi_a \times \Pi_b) \int W_{BD} \times (T_b \times \Pi_b) \int P_{2C}$
$\gamma \neq 0$	$\Pi_a \times P_{2A} \times (T_{a+b} \times \Pi_a \times \Pi_b) \int W_{BD} \times \Pi_b \times P_{2C}$
$\gamma \rightarrow \infty$	$(T_a \times \Pi_a) \int O_{2A} \times (T_{a+b} \times \Pi_a \times \Pi_b) \int W_{BD} \times (T_b \times \Pi_b) \int O_{2C}$



$\gamma$	$K$
$\gamma = 0$	$(T_a \times P_2) \int W_{ABCD}$
$\gamma \neq 0$	$P_2 \int W_{AC} \times (T_a \times P_2) \int W_{BD}$
$\gamma \rightarrow \infty$	$T_a \int O_{2A} \int W_{AC} \times (T_a \times P_2) \int W_{BD}$

$\gamma$	$K$
$\gamma = 0$	$(T_a \times P_2) \int W'_{ABCD}$
$\gamma \neq 0$	$P_2 \int W'_{AC} \times (T_a \times P_2) \int W'_{BD}$
$\gamma \rightarrow \infty$	$T_a \int O_{2A} \int W'_{AC} \times (T_a \times P_2) \int W'_{BD}$

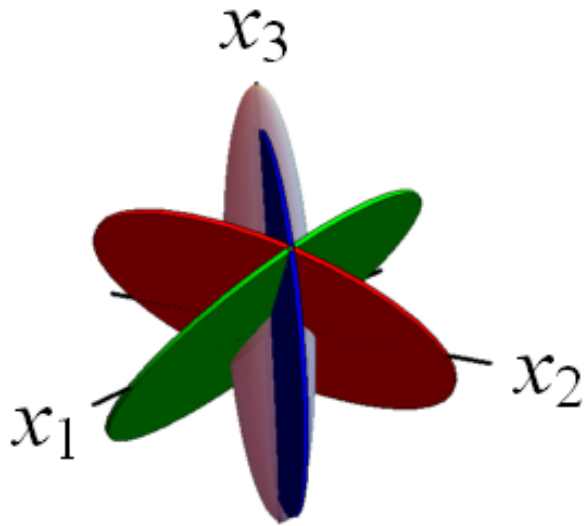
$\gamma$	$K$
$\gamma = 0$	$(T_a \times \Pi_a \times P_2) \int W_{ABCD}$
$\gamma \neq 0$	$(\Pi_a \times P_2) \int W_{AC} \times (T_a \times \Pi_a \times P_2) \int W_{BD}$
$\gamma \rightarrow \infty$	$(T_a \times \Pi_a) \int O_{2A} \int W_{AC} \times (T_a \times \Pi_a \times P_2) \int W_{BD}$

# Coxeter Model

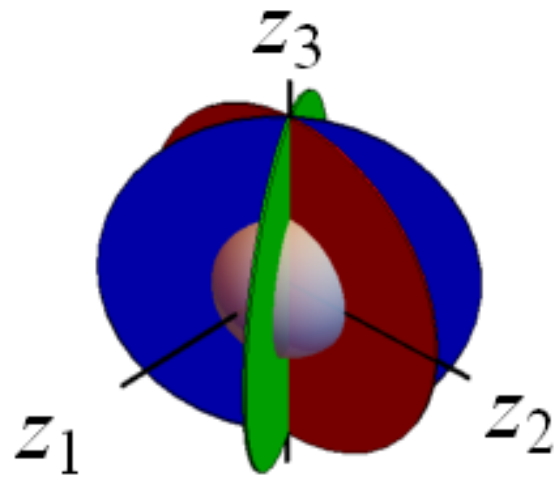
$$H = \sum_{i=1}^N \left( \frac{1}{2m_i} p_i^2 + \frac{1}{2} m_i \omega^2 x_i^2 \right) + g \sum_{\langle i,j \rangle} \delta(x_i - x_j)$$

- Same a contact interaction, but different masses with same trapping frequency
- Hard core limit  $g \rightarrow \infty$
- Solvable and (maximally super) integrable when masses have right size and right order

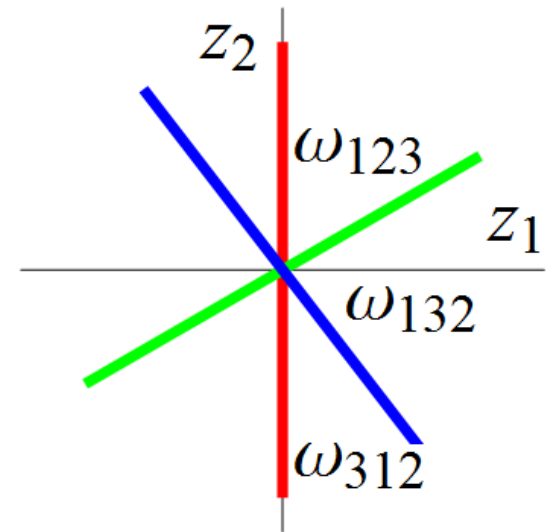
## Unequal Masses



Natural coordinates

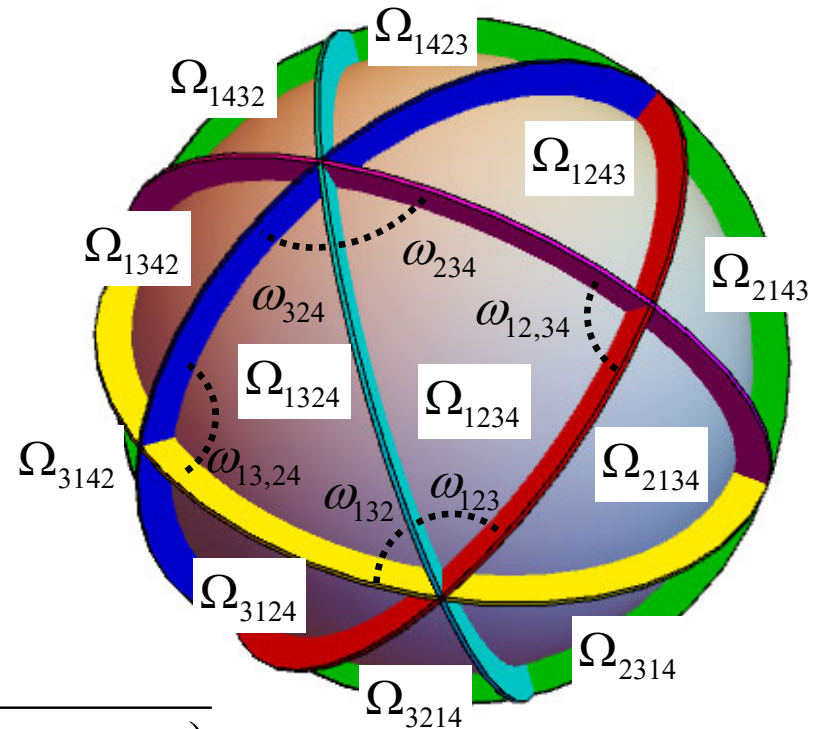
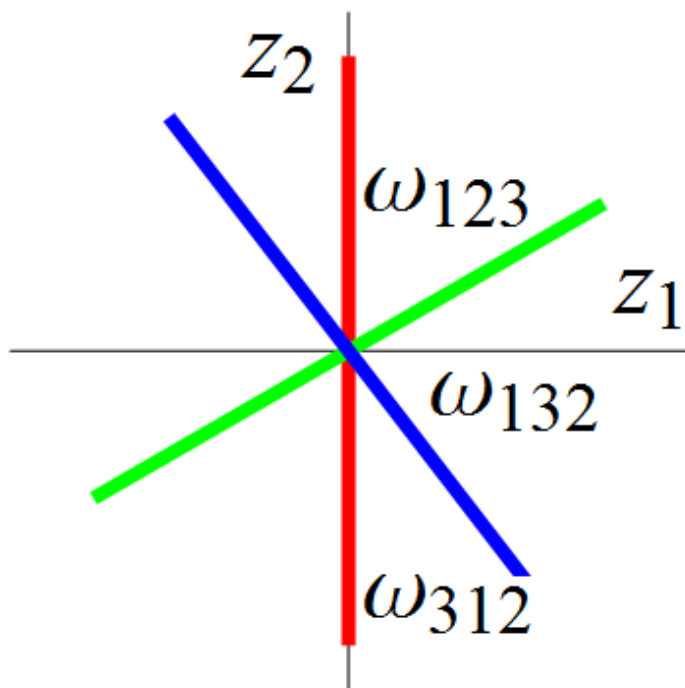


Mass-rationalized Jacobi coordinates



Relative, mass-rationalized Jacobi coordinates

$$\tan \omega_{ijk} = \sqrt{\frac{m_j (m_i + m_j + m_k)}{m_i m_k}}$$



$$\tan \omega_{ijk} = \sqrt{\frac{m_j(m_i + m_j + m_k)}{m_i m_k}}$$

$$\omega_{ij,kl} = \frac{\pi}{2}$$

Three particles: quantum billiards on a ring

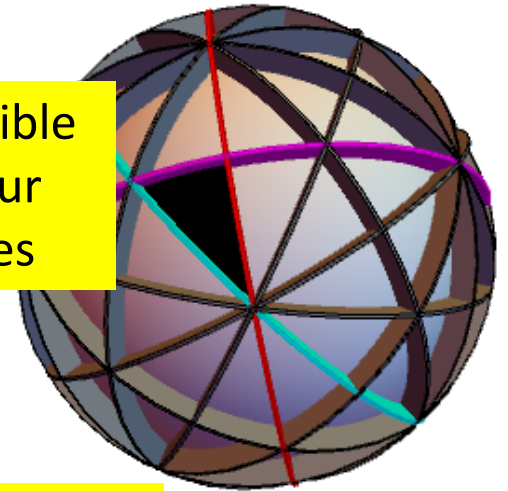
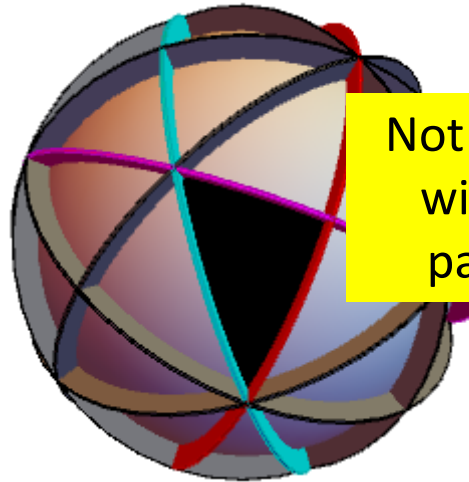
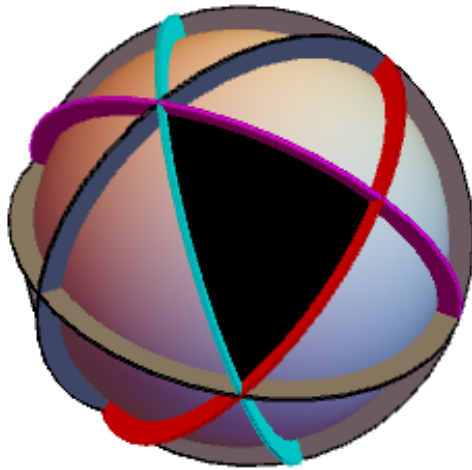
Four particles: quantum billiards in spherical triangles

Five particles: quantum billiards in 3-spherical tetrahedra

$A_3$  : tetrahedral

$C_3$  : octahedral

$H_3$  : icosahedral

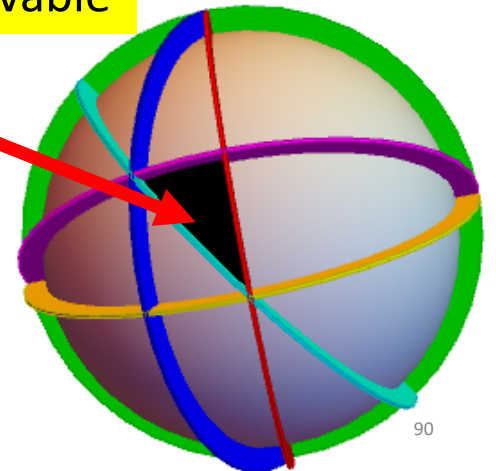
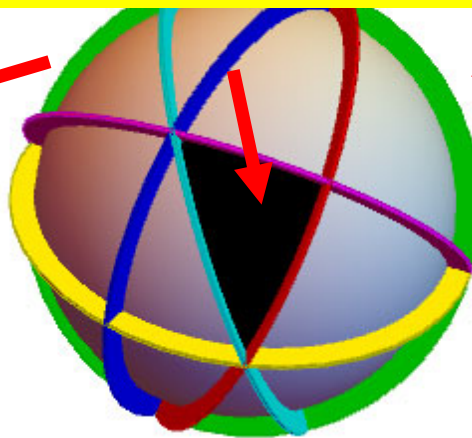
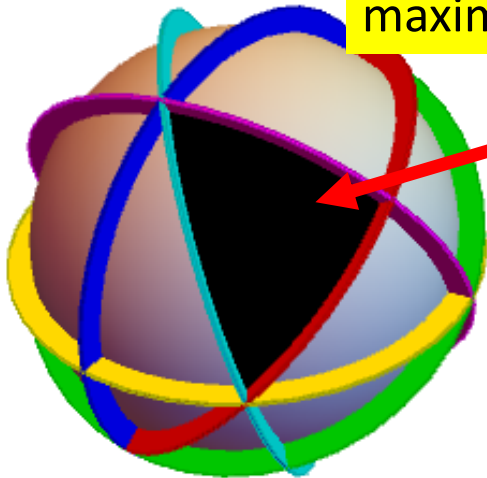


Four equal mass particles

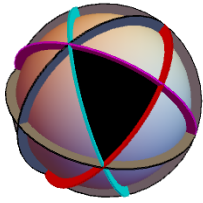
Not possible with four particles

maximally superintegrable, exactly solvable

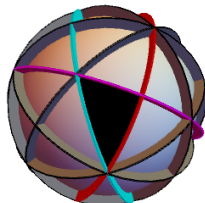
But...



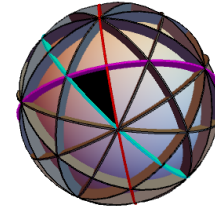
## Coxeter Groups and the Method of Images



$$A_3 \cong [3, 3]$$



$$C_3 \cong [4, 3]$$



$$H_3 \cong [5, 3]$$

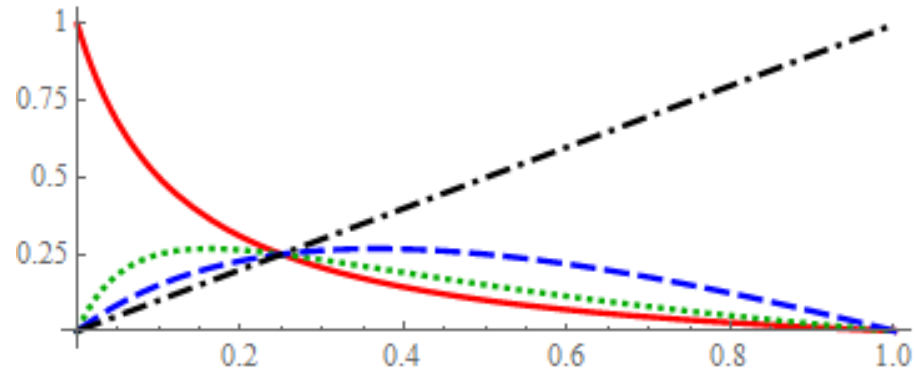
- Angles

$$[q, 3] \Rightarrow \omega_{123} = \frac{\pi}{q}, \omega_{234} = \frac{\pi}{3}, \omega_{12,34} = \frac{\pi}{2}$$

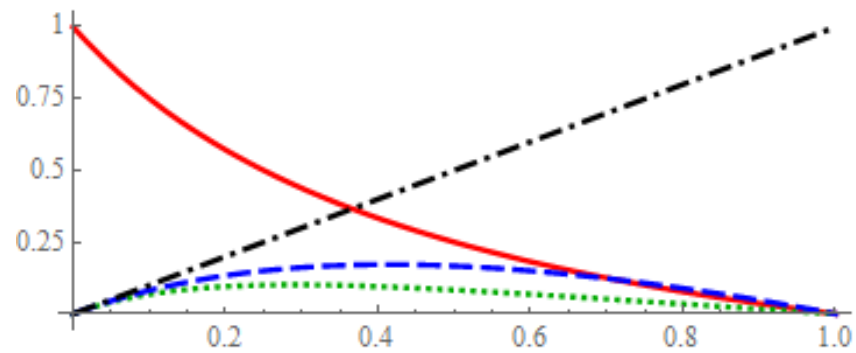
- Reflections

$$[q, 3] \Rightarrow (R_{23}R_{12})^q = (R_{34}R_{23})^3 = (R_{12}R_{34})^2 = 1$$

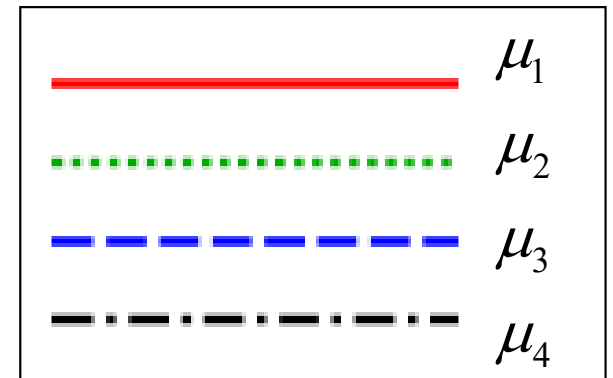
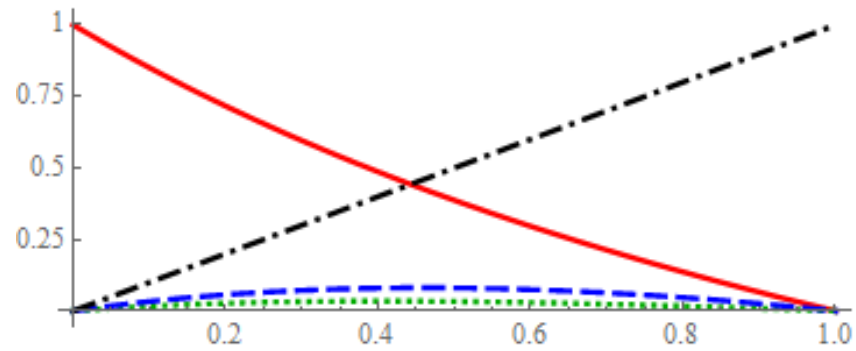
$A_3$



$C_3$



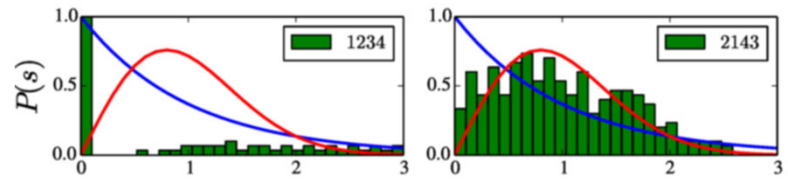
$H_3$



mass fractions

rational masses exist for A-series and C-series. Not others.

	A-series	C-series	H-type	others
	$A_2 \equiv I_2(3)$	$C_2 \equiv I_2(4)$	$H_2 \equiv I_2(5)$	$I_2(q)$
$N=$	[3]	[4]	[5]	[ $q$ ]
3	$\lambda_0 = 3$	$\lambda_0 = 4$	$\lambda_0 = 5$	
	$G = 6$			
$N=$				
4				
$N=$				
5				
$N \geq$	[ $3^{N-1}$ ]			
6	$\lambda_0 = \frac{N(N-1)}{2}$	$\lambda_0 = \frac{N(N-1)}{2}$		
	$G = (N-1)!$	$G = 2^N N!$		



algebraic solvability, separability  
 Liouville integrability, Bethe-Ansatz integrability,  
 superintegrability, maximal superintegrability,  
 ergodicity, mixing, chaos!

