Top 10 Symmetries Every Quantum Mechanic Should Know

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DAY 1

Brief Introduction to Me

- Theoretical particle physics
  - Phenomenology of resonances and decaying states
  - Relativistic quantum mechanics
- Quantum information theory
  - Relativity of entanglement
  - Entanglement in interacting particle systems
- Few-body physics
  - Symmetry, dynamics and solvable models
  - Quantum chaology
  - Topological exchange statistics and anyons

Brief Introduction to You

- Who are you?
  - Name, pronouns
- Where are you?
  - University, institute, group
- What are you doing?
  - Field, subfield, topic, subject

Pedagogical Expectations

Goals for lectures

- Survey of topics and examples
- Transfer of ideas and impressions
- Main ideas and key results
- Overview and synthesis
- Inspire you to apply symmetry methods

Not goals for lectures

- Deep dive into single subject
- Transfer of knowledge or skills
- Derivations and proofs
- Review of subfield
- Convert you to symmetry religion

Talk too fast, off-putting sarcasm, some “informality”

90-5-5

phase space

space

parameter space

configuration space

Hilbert space

Fock space

model space

operator space

geometry, symmetry, topology

metric

E.O.M.

equations of motion

Hamiltonian

probability

amplitudes, densities, currents

C.S.C.O

complete set of commuting observables

action

Lindbladian

Some types of symmetries

Geometric:
Symmetries that map a space onto itself. Could be physical space, configuration space, phase space, parameter space,…

Kinematic:
Symmetries that commute with Hamiltonian

Dynamic:
Symmetries that act on algebras of operators that includes Hamiltonian and/or preserve the action

Top 10 Symmetries
Every Quantum Mechanic Should Know

Fundamental symmetries

- Involution
- Products of Involutions

Geometrical Symmetries

- Point symmetry
- Rotational symmetry
- Translational symmetry
- Euclidean symmetry
- Lattice symmetry

More general kinematic and dynamics symmetries

- Permutation symmetry
- Unitary symmetry
- Galilean symmetry

Non-relativistic

For each symmetry

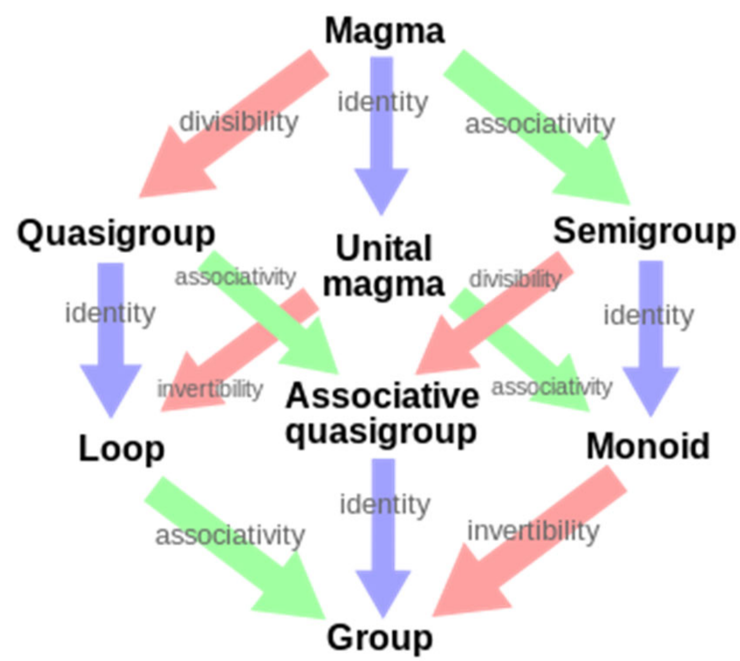
- Realizations in quantum theory
- Example group
  - Group structure
  - Group representations
  - Group classification
- Applications
- Relations to other symmetries and groups

Next 8 best symmetries

- Symplectic symmetry
- Conformal symmetry
- Exchange symmetries
- Lorentz symmetry
- Poincare symmetry
- Gauge symmetry
- Schrodinger symmetries
- Oscillator symmetries

Group definition

- A set with binary operation that satisfies the group axioms
  - Closure: product in the set
  - Associative: abc = (ab)c = a(bc)
  - Identity: unique element is the identity
  - Inverse: every element has unique inverse



Top 10 Groups Every Physicist Should Know

Discrete

Finite:

1. Z_2
2. V_4
3. S_3

4. Z

Continuous

5. R

Compact:

6. $U(1)$
7. $SO(3)$

8. $SL(2,\mathbb{R})$
9. $G(3,1)$
10. $P(3,1)$

Key applications of group theory to symmetry in quantum theory

1. Algebraic solvability and harmonic analysis
2. Classification of possible symmetries
3. Selection rules
4. Spectroscopic labels and degeneracy
5. Numerical efficiency and symmetry-adapted bases
6. Invariance, invariants, and covariance
7. Indistinguishable particles
8. Symmetry breaking and phase transitions
9. Integrability and chaos
10. Topological control

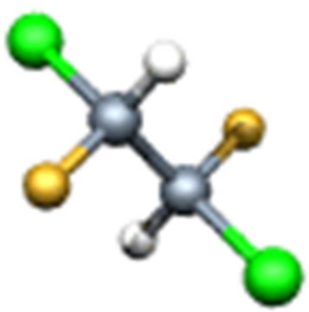
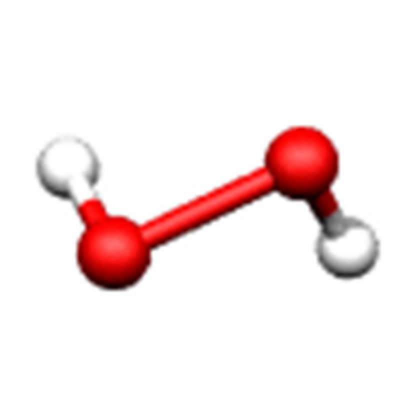
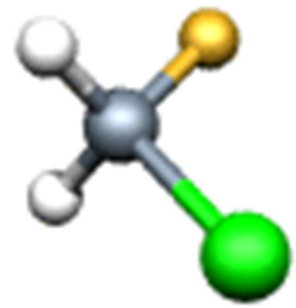
1. Involution

Yin and yang, baby

Realizations

- Parity
  - 1D: Reflection symmetry
  - 2D: Rotation by π symmetry
  - 3D: Inversion symmetry
- Time reversal symmetry
- Charge conjugation symmetry
- CPT symmetry
- Chiral aka sublattice symmetry
- Symmetric group on two objects

$$Z_2 \sim C_2 \sim D_1 \sim S_2 \sim A_1 \sim O(1)$$

https://www.staff.ncl.ac.uk/j.p.goss/symmetry/Molecules_pov.html

Example Group: Z_2

Key math concepts: presentation, group table, realization, representation, induced representation on a function space, unitary representation

Induced Representations of Parity in 1D

- Representation on space $\mathbb{R}$

$$\Pi x = x' = -x \qquad \Pi^2 = 1$$

- Induced representation on Hilbert space $\mathcal{H} = L^2(\mathbb{R})$

$$\hat{U}(\Pi)\psi(x) = \psi'(x) = \psi(-x)$$

- Furnishes a unitary representation on $\mathcal{H}$

$$\hat{U}(\Pi) = \hat{U}(\Pi)^\dagger \qquad \hat{U}(\Pi)\hat{U}(\Pi) = \hat{\mathbb{I}} \qquad \hat{U}(\Pi) \equiv \hat{\Pi}$$

Induced Representations of Parity in 1D

- Projection operators

$$\hat{P}_+ \equiv \frac{1}{2}(\hat{\mathbb{I}} + \hat{\Pi}) \qquad \hat{P}_- \equiv \frac{1}{2}(\hat{\mathbb{I}} - \hat{\Pi})$$

- Decompose Hilbert space

$$\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_- \qquad \langle \hat{P}_+\psi | \hat{P}_-\psi \rangle = 0$$

- Two representations

$$\hat{\Pi}\mathcal{H}_+ = \mathcal{H}_+ \qquad \hat{\Pi}\mathcal{H}_- = \mathcal{H}_-$$

Parity as a Kinematic Symmetry

- Hamiltonian provides decomposition of Hilbert space into energy sectors

$$\mathcal{H} = \bigoplus_{E \in \sigma} \mathcal{H}_E$$

- Commutes with Hamiltonian

$$\widehat{\Pi}\widehat{H} = \widehat{H}\widehat{\Pi} \qquad \widehat{\Pi}\psi_E(x) = \pm\psi_E(x)$$

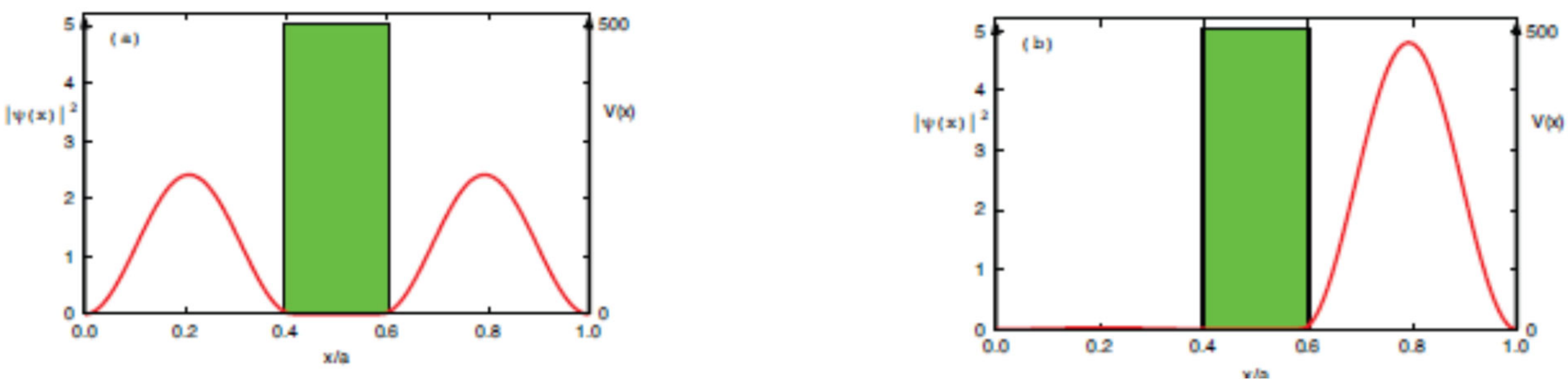
- Spectral decomposition

$$\sigma = \sigma_+ \bigoplus \sigma_- \qquad \mathcal{H} = \left(\bigoplus_{E \in \sigma_+} \mathcal{H}_E\right) \oplus \left(\bigoplus_{E \in \sigma_-} \mathcal{H}_E\right)$$

Why *NOT* Symmetry?

FRAGILE

the elephant and the flea



T Dauphinee, F Marsiglio, AJP (2015)

Related Symmetries and Groups

Cyclic groups

2. Product of involutions

Deceptively simple, but they classify fundamental theories

Realizations

COMBINE!

- Parity
  - 1D: Reflection symmetry
  - 2D: Rotation by π symmetry
  - 3D: Inversion symmetry
- Time reversal symmetry
- Charge conjugation symmetry
- CPT symmetry
- Chiral aka sublattice symmetry
- Symmetric group on two objects

$$V_4 = Z_2 \times Z_2$$

Key math concepts: direct product of groups, order of group, subgroup, abelian vs non-abelian, faithful vs unfaithful representation, antiunitary representation, projective representation

Representations of V_4

- Finite dimensional representations

$$D: V_4 \to GL(N, \mathbb{C}) \text{ such that}$$
$$g_1 g_2 = g_3 \to D(g_1)D(g_2) = D(g_3)$$

- One-dimensional representations
  - Abelian
  - Can be chosen to be unitary

    $$D(g^{-1}) = D^{\dagger}(g)$$
  - Not faithful

	e	a	b	ab
A_1	1	1	1	1
A_2	1	1	1	-1
B_1	1	-1	1	-1
B_2	1	1	-1	-1

- Higher dimensional unitary representations are direct sums

Wigner’s Theorem (1931)

Definitions:

- A symmetry transformation maps states into states and preserves probabilities
  - Pure states are rays in a Hilbert space
- Symmetry transformations of a physical system form a group

Theorem:

- All symmetry transformations can be represented as unitary or antiunitary operators on a Hilbert space.
- The Hilbert space carries a projective representation of the group of symmetry transformations.

Projective Representations

- Wigner 1: Corresponding to a symmetry transformation of physical states is a linear and unitary (or antiunitary) transformation on vectors

$$\left|\langle\psi|\phi\rangle\right|^2 = \left|\langle\psi'|\phi'\rangle\right|^2 = \left|\langle T\psi|T\phi\rangle\right|^2$$

- Wigner 2: If there is a group of symmetries, group representation can be projective, i.e. like $T(R_1)T(R_2) = \omega(R_1, R_2)T(R_1R_2)$

- Wigner 3: Projective reps of a simply connected group without central charges are equivalent to (single-valued, true) reps.
- Wigner 4: Projective reps of a group are equivalent to the true reps of the universal covering group of that group.

Weyl, Bargmann may have helped.

Representations of V_4

- Projective representations

$$D: V_4 \to GL(N, \mathbb{C}) \text{ such that}$$
$$g_1 g_2 = g_3 \to D(g_1)D(g_2) = \omega(g_1, g_2)D(g_3)$$

	e	a	b	ab
A_1	1	1	1	1
A_2	1	1	1	-1
B_1	1	-1	1	-1
B_2	1	1	-1	-1
E	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

Representations of V_4

- Antiunitary representation example:

$$T \to e^{i\pi S_y} K$$

  - Spin-1/2 system

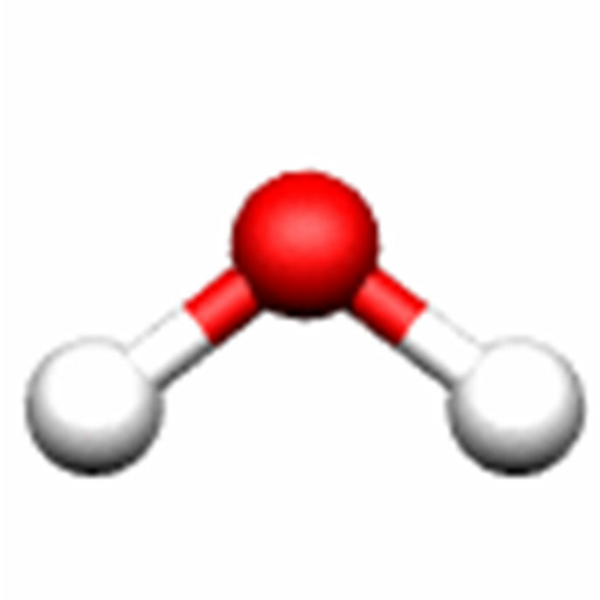
$$T \to \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} K$$

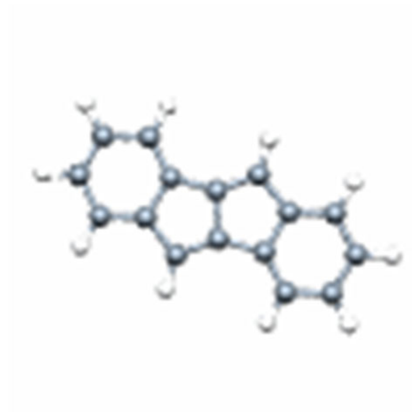
  - Spin-*j* system

$$T^2 = (-1)^{2j}$$

Applications

- Geometrical and kinematic symmetry
- C-P-T classification of particles and interactions
- Altland-Zirnbauer "ten-fold way" C-T-S classification of symmetric manifolds, random matrices, and topological insulators



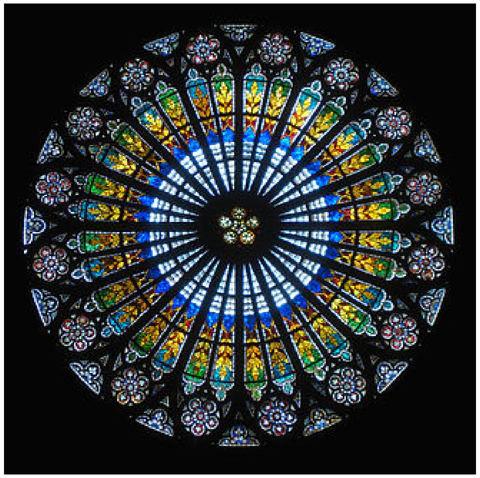


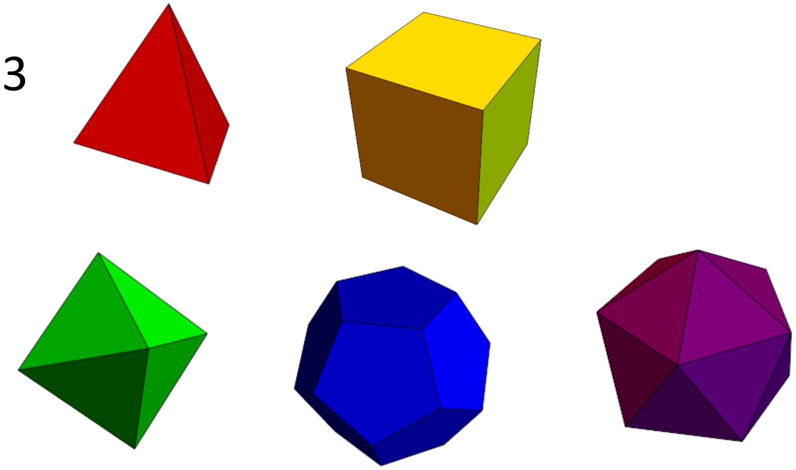
https://www.staff.ncl.ac.uk/j.p.goss/symmetry/

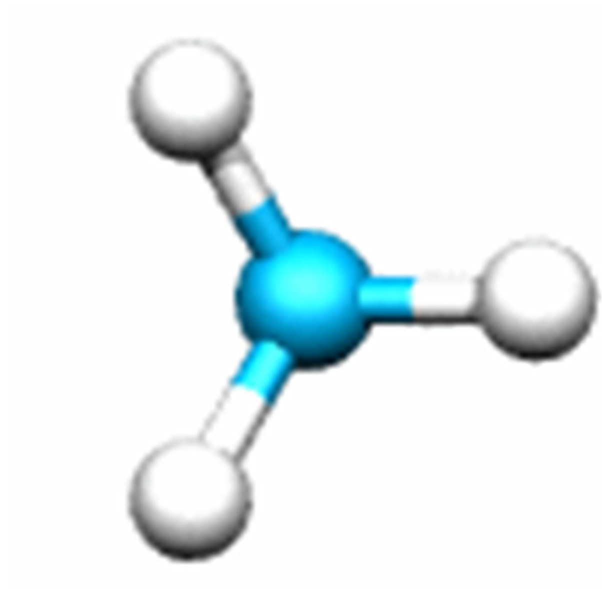
3. Point symmetry

Euclidean geometry and point groups

- Convex regular polytopes:
  - 1D: 1
  - 2D: ∞ (rosette groups)
  - 3D: 5
  - 4D: 6
  - >4D: only 3







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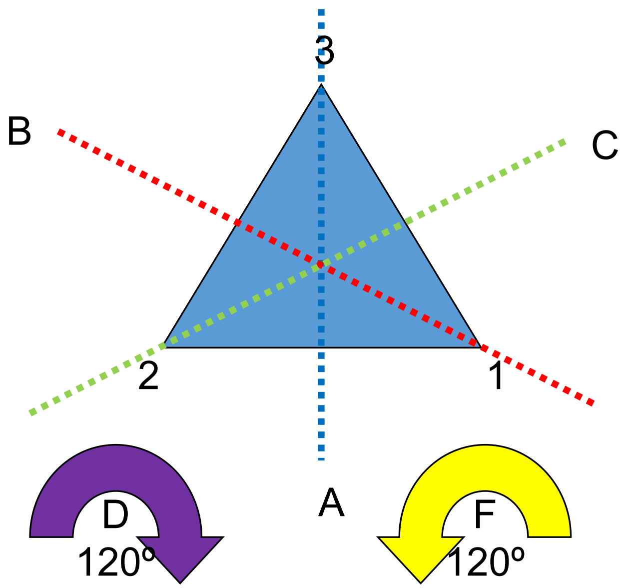
Example Group: C_{3v}

Key math concepts: conjugacy classes, irreducible representations, characters, character tables, symmetry breaking

Take a second…



	E	A	B	C	D	F
E	E	A	B	C	D	F
A	A	E	D	F	B	C
B	B	F	E	D	C	A
C	C	D	F	E	A	B
D	D	C	A	B	F	E
F	F	B	C	A	E	D

$$C_{3v} \sim D_3 \sim S_3 \sim A_2$$

Representations of C_{3v}

Character tables

C_{3v}	$e(1)$	$\sigma_v(3)$	$C_3(2)$
A_1	1	1	1
E	2	0	-1
A_2	1	-1	1

Irreducible representations (irreps)

- An irreducible representation has no subspace invariant under all matrices in the representation.
- Inequivalent irreps are orthogonal when summed over the group.
- Every finite-dimensional unitary representation on a Hilbert space is completely reducible into the direct sum of irreducible representations.
- Every irreducible representation of a finite group can be chosen to be unitary.
- For finite group, number of inequivalent irreducible representations equal to number of conjugacy classes.
- For finite group, the regular representation has as many copies of each irrep as the dimension of the irrep.

Irrep and character facts

- Dimensions of irreps $$\sum_{\mu} n_{\mu}^{2} = [G]$$
- Orthogonality of irrep characters

$$\sum_{i} [g_i] \chi^{(\mu)*}(g_i) \chi^{(\nu)}(g_i) = [G] \delta_{\mu\nu}$$

- Useful special case $$\sum_{i} [g_i] \left| \chi^{(\mu)}(g_i) \right|^2 = [G]$$

Schur’s Second Lemma

- Lemma: The only matrix that commutes with all elements of an irreducible representation of a group is a multiple of the identity.
- Lemma originally proven for finite groups but can be generalized.
- Can be used to prove that abelian groups only have one-dimensional irreps.
- Basis of the degeneracy theorem

Degeneracy Theorem

- Every energy eigenspace must carry a representation of a symmetry group of the Hamiltonian.
- Special case: all the energy eigenspaces carry irreducible representations of the symmetry group of the Hamiltonian.
  - Then we the *kinematic symmetries* of the Hamiltonian are sufficient to explain the degeneracies.
- Unexplained degeneracies?
    - We could be missing a kinematic symmetry…keep looking!
    - Or…there could be dynamic symmetries.
    - Or there could be accidental symmetries. ???

Kinematic Symmetry

$$\mathrm{K} \Rightarrow \quad U(g)H = HU(g) \ \forall \ g \in \mathrm{K}$$

$\left\{ \text{Invariant operators} \right\} \Leftrightarrow \left\{ \text{Conserved quantities} \right\}$

Goal: find maximal* kinematic symmetry group of Hamiltonian

$\left\{ \text{Projective irreducible representations} \right\} \Leftrightarrow \left\{ \text{Energy eigenspaces} \right\}$

Finite group theorems

- Lagrange's theorem: The order of every subgroup divides the order of the group.
- Cayley's theorem: Every subgroup is isomorphic to permutation subgroup of group.
- Maschke's theorem: Every representation of a finite group can be written as a direct sum of irreducible representations
- Second cohomology group classifies projective representations and whether there is a doubling due to time reversal symmetry

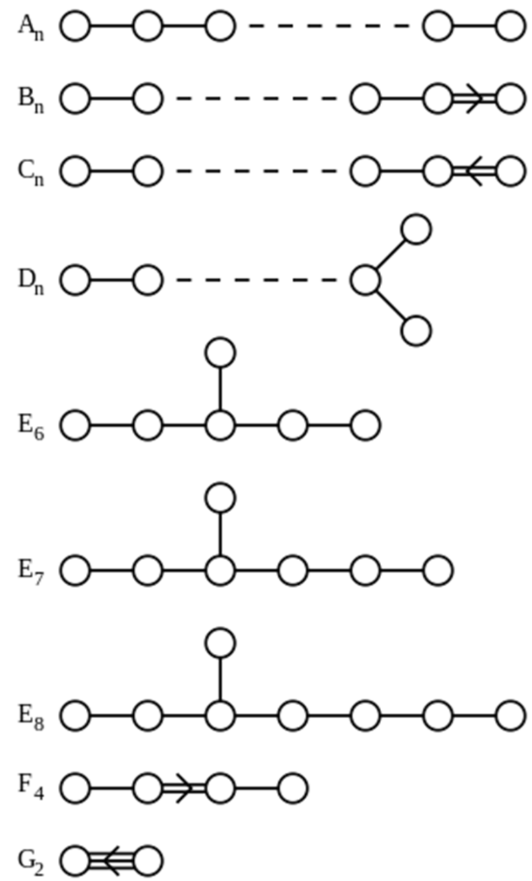
4. Rotational symmetry

Isotropy

SO(3)

Key math concepts: Lie group, Lie algebra, covering group, direct product of representations, Clebsch-Gordan coefficients

Interlude:
Lie groups and Lie algebras



Parameterizing SO(3)

- Right-handed Euler angle parameterization

moving axes

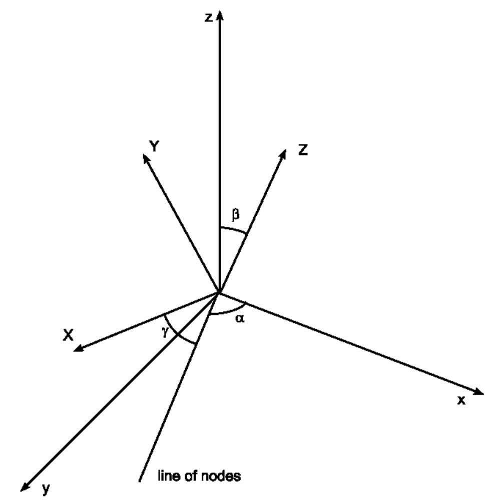
$$R(\alpha,\beta,\gamma) = R_{z''}(\gamma)R_{y'}(\beta)R_z(\alpha)$$

fixed axes

$$= R_z(\alpha)R_y(\beta)R_z(\gamma)$$

$$R_z(\phi) = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad R_y(\theta) = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$$

– Ranges: $-\pi \le \alpha < \pi, 0 \le \beta < \pi, -\pi \le \gamma < \pi$



Other parameterizations

- zxz-Euler, zyx-Euler
- direction cosines

Three-parameter compact Lie group $R \in SO(3)$

- Representation on space $\mathbb{R}^3$

$$R\mathbf{x} = \mathbf{x}' \qquad R^T R = R R^T = I_3 \qquad \det(R) = 1$$

- Induced representation on Hilbert space $\mathcal{H} = L^2(\mathbb{R}^3)$

$$\hat{U}(R)\psi(\mathbf{x}) = \psi'(\mathbf{x}) = \psi(R^{-1}\mathbf{x})$$

$$\hat{U}^{-1}(R) = \hat{U}^{\dagger}(R) = \hat{U}(R^{-1}) \qquad \hat{U}(R_2 R_1) = \hat{U}(R_2)\hat{U}(R_1)$$

Irreducible representations

- Hilbert space decomposed into sectors

$$\mathcal{H} = \bigoplus_{\ell \in \mathbb{Z}_+} \mathcal{H}_\ell \qquad \mathcal{H}_\ell = \mathcal{D}_\ell \times \mathcal{V}_\ell \qquad \mathcal{V}_\ell = \mathbb{C}^{2\ell+1}$$

- Rotations leave irrep invariant $SO(3) \supset SO(2)$

$$U(R)\left|\ell m\right\rangle = \sum_{m'} \left|\ell m'\right\rangle D^{(\ell)}_{m'm}(R) \qquad \hat{U}(R) Y_{\ell m}(\theta,\phi) = \sum_{m'} Y_{\ell m'}(\theta,\phi) D^{(\ell)}_{m'm}(R)$$

- Orbital angular momentum is a good quantum number

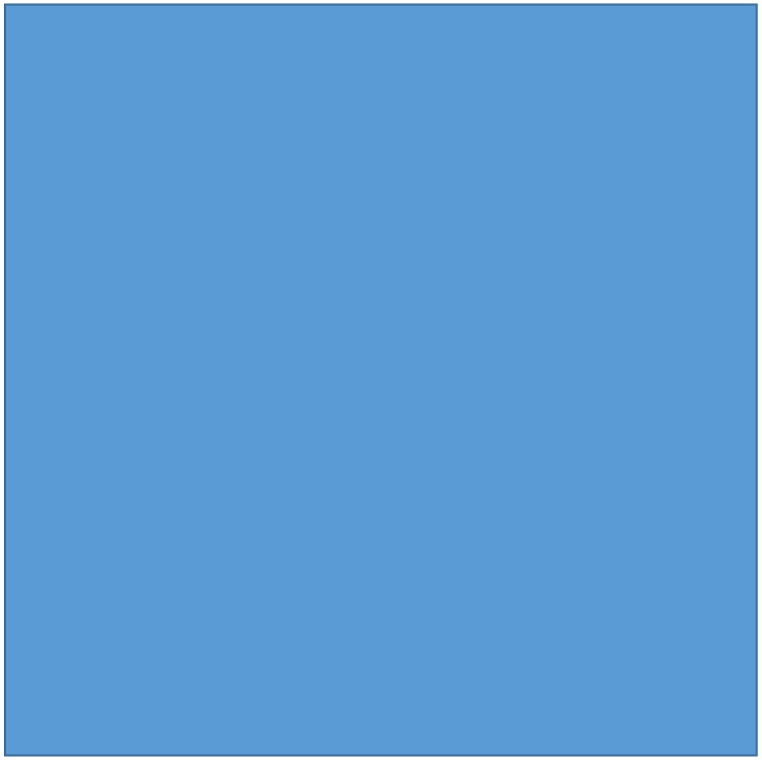
$$\hat{U}(R)\hat{H} = \hat{H}\hat{U}(R) \qquad \psi_{E\ell m}(\mathbf{x}) = R_{E\ell}(r) Y_{\ell m}(\theta,\phi)$$

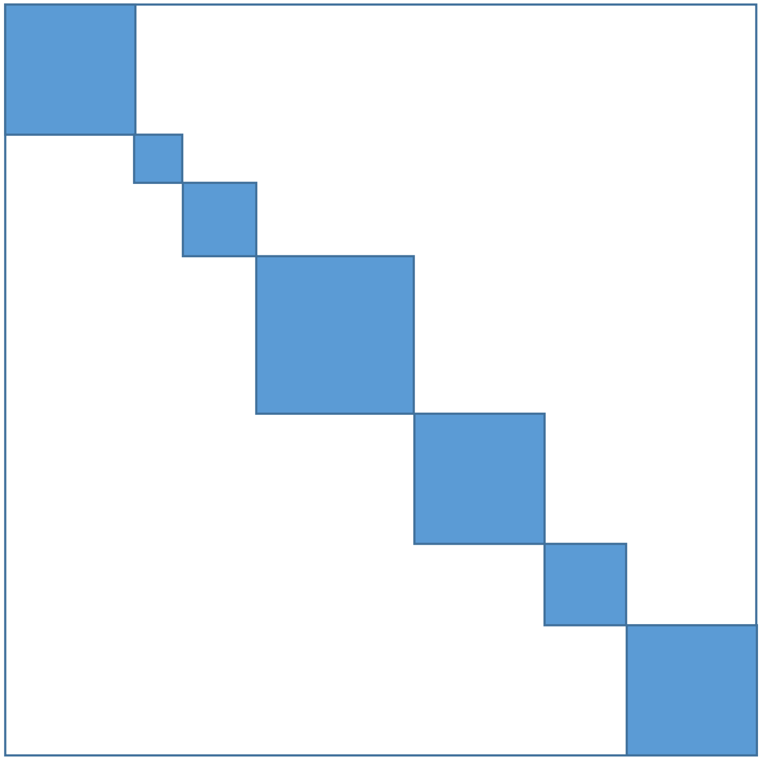
Clebsch-Gordan Coefficients

Selection rules

- A matrix element $\langle\psi|O|\phi\rangle$ is identically zero if the direct product representation $D^{\psi}\otimes D^{O}\otimes D^{\phi}$ does not contain the "trivial" totally symmetric irreducible representation.
- Special cases: No operator, symmetric operator
- Key application: Wigner-Eckart Theorem

Symmetry and Exact Diagonalization





Every zero is a missed symmetry….   …but maybe that is OK Computer

Manifold SO(3)

- Isomorphic manifolds: $-\pi \leq \alpha < \pi, 0 \leq \beta < \pi, -\pi \leq \gamma < \pi$
  – Open ball of radius π, such that opposite points are identified
  – S_3, 3-sphere in 4D, with opposite points identified
- Not simply connected

Universal Covering Group

- UCG $\tilde{G}$ of Lie group G is a unique, simply connected Lie group with a homomorphism $\tilde{G} \rightarrow G$ and isomorphic Lie algebras.

$\mathrm{SU}(2)$ is UCG of $\mathrm{SO}(3)$

$$\mathrm{SO}(3) \simeq \mathrm{SU}(2)/Z_2$$

- UCG of SO(*n*) sometimes called Spin(*n*)

Applications of SO(3)

- Single particle in three dimensions, free or spherical trap
- Particle on a sphere
- Relative motion of two particles in three dimensions and Galilean invariant interaction, free or harmonic trap, e.g. Hydrogen
- Configuration space of three non-interacting particles in one dimension, free or harmonic trap
- Relative configuration space of four non-interacting particles in one dimension, free or harmonic trap

Generalize to other dimensions

Dimensionality

D\N	1	2	3	4	5	6	7	8
1	0	0	1	2	3	4	5	6
2	0	0	2	4	6	8	10	12
3	0	0	2	6	9	12	15	18

- Separate internal discrete degrees of freedom $(\mathbb{C}^m)^{\otimes N} = \mathbb{C}^{Nm}$
- Separate center of mass DOF for quadratic traps
- Separate relative hyperradial DOF for certain traps, interactions
  - Remaining DOFs form a sphere
- Remove orientation: shape space

Other schemes to separate DOF:
adiabatic, Born-Opp, s-waves

Generalizations

- O(3)
  - Rotations
  - Reflections and rotoreflections, including parity
- SO(4)
  - Bound states of hydrogen atom
  - Two subgroup chains
- O(N)
  - Hyperspherical harmonics

$$\det O = 1$$

$$\det O = -1$$

$$SO(4) \supset SO(3) \supset SO(2)$$

$$SO(4) = SO(3) \times SO(3)$$

Hyperspherical Harmonics Subgroup Chain

$$O(5) \supset O(4) \supset O(3) \supset O(2) \supset O(1)$$

$O(1): \mu_1 = +1, -1 \qquad d(\mu_1) = 1$

$O(2): \mu_2 = 0, 1, 2, 3, \ldots \quad d(0) = 1, d(\mu_2 > 0) = 2$

$O(3): \mu_3 = 0, 1, 2, 3, \ldots \quad d(\mu_3) = 2\mu_3 + 1$

$O(4): \mu_3 = 0, 1, 2, 3, \ldots \quad d(\mu_4) = \left(\mu_4 + 1\right)^2$

$O(5): \mu_4 = 0, 1, 2, 3, \ldots \quad d(\mu_5) = \sum_{\mu_4=0}^{\mu_5} \mu_4^2 = (2\mu_5 + 1)(\mu_5 + 1)\mu_5 / 6$

Relative motion of hydrogen atom

- SO(3) kinematic symmetry provides CSCOs and state labels

$$\{H, L^2, L_3\}$$

$$H|nlm\rangle = \left(-E_1 / n^2\right)|nlm\rangle$$
$$L^2|nlm\rangle = \hbar^2 l(l+1)|nlm\rangle$$
$$L_3|nlm\rangle = \hbar m|nlm\rangle$$

- Actually, SO(4) kinematic symmetry explains degeneracy of bound states; superintegrable and multiseparable

$$\{H, A_3, L_3^2\} \quad \{H, A_3 + aL^2, L_3^2\} \quad \{H, L^2, L_1^2 + fL_3^2\}$$

spherical coordinates
parabolic rotational coordinates
shifted spheroidal coordinates
spheroconical coordinates

- SO(4,2) dynamic symmetry provides selection rules and matrix elements